E621: The linear Stark effect.

Submitted by: Ilya Bukchin

The problem:

A Hydrogen atom is positioned in a constant electric field $\mathcal{E}_0\hat{z}$.
As a result , the degeneracy at the first exited energy level (four dimentional sub-space) is removed.
What are the Energy levels and the Eigenvalues if we ignore the other energy levels?
Write an Expression for the first order of the perturbing Hamiltonian.

The solution:

First we define the Hamiltonian for the Hydrogen atom, A Hydrogen atom is a particle of charge e and reduced mass M that is positioned in a potential: $V_0 = -\frac{\alpha}{r}$. After solving for each l get : $E_{\nu,l,m} = -\frac{\alpha^2 M}{2(l+\nu)^2}$.
The Hamiltonain that describes the first exited level ($n = l + \nu = 2$) In the basis :

$$|n,l,m\rangle \to |200\rangle, |210\rangle, |211\rangle, |21-1\rangle$$

Is:

$$\mathcal{H}_0^{(n=2)} = -\frac{\alpha^2 M}{8}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The perturbing potential for this problem is : $V = -e\mathcal{E}_0 z = -e\mathcal{E}_0 r\cos\theta$
Therefore, the new Hamiltonian is :

$$\mathcal{H} = \mathcal{H}_0 - e\mathcal{E}_0 r\cos\theta \tag{1}$$

Now we will calculate the perturping potential:

$$V_{n',l',m',n,l,m} = -e\mathcal{E}_0\langle n'l'm'|r\cos\theta|nlm\rangle = -e\mathcal{E}_0\int_0^\infty R^*_{n',l'} r^3 R_{n,l} dr \int_0^{4\pi} Y^*_{l'm'} Y_{lm} cos(\theta) d\Omega \tag{2}$$

Where:

$$\langle r\theta\phi|nlm\rangle = R_{n,l}Y_{l,m} \tag{3}$$

And in paticular :

$$n = 2 \tag{4}$$

$$R_{20} = \left(\frac{M\alpha}{2}\right)^{\frac{3}{2}} \frac{rM\alpha}{\sqrt{3}} e^{-\frac{rM\alpha}{2}} \tag{5}$$

$$R_{21} = \left(\frac{M\alpha}{2}\right)^{\frac{3}{2}} (2 - rM\alpha)\, e^{-\frac{rM\alpha}{2}} \tag{6}$$

$$Y_{00} = \sqrt{\frac{1}{4\pi}} \tag{7}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta \tag{8}$$

From now on We will omit n.

Now we can write the perturbing electric field as follows : $V = -\sqrt{\frac{4\pi}{3}} e\mathcal{E}_0 Y_{10}$.
(substituting $cos(\theta)$ for $\sqrt{\frac{4\pi}{3}} Y_{10}$).
Then , equation (2) becomes :

$$V_{l',m',l,m} = -\sqrt{\frac{4\pi}{3}} e\mathcal{E}_0 \int_0^\infty R^*_{2,l'} r^3 R_{2,l} dr \int_0^{4\pi} Y^*_{l'm'} Y_{lm} Y_{10} d\Omega \tag{9}$$

And then , from parity considerations we see that the only case in which the expression is nonzero is when the function is odd,
meaning : $l' \neq l$ and $m' = m = 0$
and because for $m = 0$, Y_{lm} and R_{nl} are Real , the expression is identical for $l = 1$ and $l = 0$
Then : subsituting Y_{00} for $\frac{1}{\sqrt{4\pi}}$ and recalling the normlization of Y_{10} we get right away the radial integral:

$$V_{1,0,0,0} = V_{0,0,1,0} = -\frac{e\mathcal{E}_0 (M\alpha)^4}{24} \int_0^\infty r^4 \left(2 - rM\alpha\right) e^{-rM\alpha} \tag{10}$$

$$\left\{ I_n = \int_0^\infty r^n e^{-r} : I_n = -nI_{n-1} : I_0 = 1 \right\} \tag{11}$$

$$= ... = \frac{3e\mathcal{E}_0}{M\alpha} \tag{12}$$

$$V = \frac{3e\mathcal{E}_0}{M\alpha} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now we Diagonalize $\mathcal{V}$ in order to find the first order energy correction.

$$V = \frac{3e\mathcal{E}_0}{M\alpha} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

And the new energy levels are :

$$E_1 = -\frac{\alpha^2 M}{8} + \frac{3e\mathcal{E}_0}{M\alpha}$$

$$E_2 = -\frac{\alpha^2 M}{8} - \frac{3e\mathcal{E}_0}{M\alpha}$$

$$E_3 = -\frac{\alpha^2 M}{8}$$

$$E_4 = -\frac{\alpha^2 M}{8}$$

With the corresponding eigenstates

$$|E_1\rangle = \frac{1}{\sqrt{2}} \left(|210\rangle + |200\rangle\right) \tag{13}$$

$$|E_2\rangle = \frac{1}{\sqrt{2}} \left(|210\rangle - |200\rangle\right) \tag{14}$$

$$|E_3\rangle = |211\rangle \tag{15}$$

$$|E_4\rangle = |21-1\rangle \tag{16}$$