E6150: symmetry breaking perturbation, degeneracies removal

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The problem:

A particle with spin 1 described by the hamiltonian :

$$\mathcal{H} = 2\hat{S}_x^2 - 4\hat{S}_y^2 + 6\hat{S}_z^2 + 1 + hS_z$$

for the first three questions, assume $h = 0$.
we define the projector operators as $P_i = \hat{1} - S_i^2$.

(1) on which base does the projection (for example)P_z project?
(2) Find the eigenstates of the Hamiltonian (and write them in the standard representation).
(3) what are the eigenenergies?

a magnetic field, h, is turned on in the Z direction. for simplicity, we've absorbed all the coefficients into the definition of h.

(4) write the interaction element in the standard basis.
(5) write the new hamiltonian in the standard basis.
(6) What are the eigenenergies of this new hamiltonian?
(7) plot these energies as a function of the parameter h.

the particle was prepared in a state of linear polarization in the z axis.
the magnetic field was turned on adiabatically so that eventually $h \gg 1$.
due to problems in the lab the magnetic field was slightly skewed from the horizontal direction.

(8) what is the state of the spin in the end of the adiabatic process?

The solution:

(1) we shall write P_x, P_y and P_z according to the given formula: $P_i = \hat{1} - S_i^2$

$$P_z = \hat{1} - S_z^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

thus, we can easily see that this is the projector on the $|e_z\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ state.

following the same method, we shall find that :

$$P_y|e_x\rangle = |e_x\rangle$$

$$P_x|e_y\rangle = |e_y\rangle$$

we can see that the projectors operators now project the linear basis.
so the hamiltonian may be written as:

$$\mathcal{H} = 5 - 2P_x + 4P_y - 6P_z$$

(2) we can see from the above hamiltonian that the eigenstates of the Hamiltonian are:

$$|e_x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}(|\Downarrow\rangle - |\Uparrow\rangle)$$

$$|e_y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}i\\0\\i\end{pmatrix} = \frac{i}{\sqrt{2}}(|\Downarrow\rangle + |\Uparrow\rangle)$$

$$|e_z\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix} = |\Updownarrow\rangle$$

(3) we shall operate the diagonalized hamiltonian on the eigenvalues:

$H|e_i\rangle = \hat{5}|e_i\rangle - 2P_x|e_i\rangle + 4P_y|e_i\rangle - 6P_z|e_i\rangle = E_i|e_i\rangle$

such that we get:

$$H|e_x\rangle = 9|e_x\rangle$$

$$H|e_y\rangle = 3|e_y\rangle$$

$$H|e_z\rangle = -1|e_z\rangle$$

now the magnetic field is turned on

(4) in the standard basis S_z is represented by the matrix $\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$. so that obviously, the new interaction element, should be:

$V = h\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$

(5) $H = H_0 + V$ so, H_0 is diagonal in the linear basis. so we only need to represent V in the same basis. we shall do this by multiplying the $V = hS_z$ matrix by each of the linear basis vectors, and representing the result in the linear basis:

$$V|e_x\rangle = i|e_y\rangle$$

$$V|e_y\rangle = -i|e_x\rangle$$

$$V|e_z\rangle = 0$$

so that we get: $V = h\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}$ and, finally, we get:

$H = \begin{pmatrix}9&-ih&0\\ih&3&0\\0&0&-1\end{pmatrix}$

(6,7) we shall simply find the eigenvalues of the H matrix, thus finding:

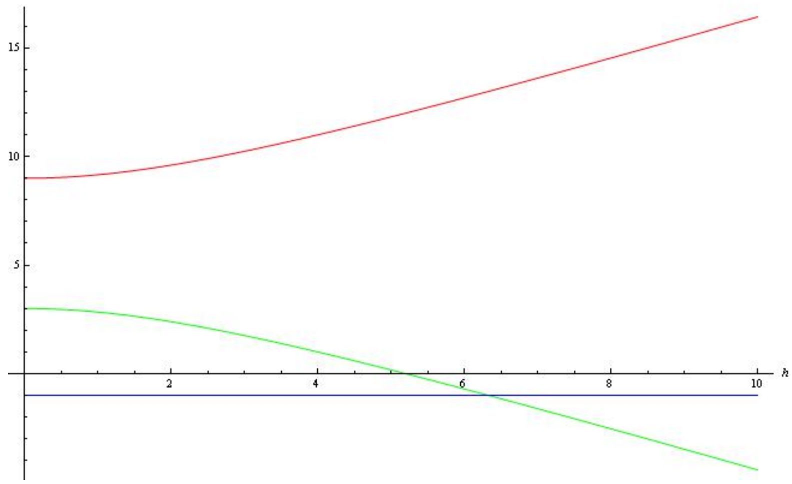

Figure 1: *black* : E_3, *purple* : E_1, *green* : E_2

$E_{1,2} = 6 \pm \sqrt{9+h^2}$
$E_3 = -1$

(8) the particle is set up in the $\hat{z}$ direction, so we know that it's in the eigenstate that is the hamiltonians ground state. because the setup is not perfect, the particle has a slight tilt towards the other states too, and so, when the magnetic field is enlarged, it's state may change and it may also come to a point where it has a degeneracy with another eigenenergie. for the adiabatic condition, we can see by taking $h \Rightarrow \infty$ that the states must change so that the state who's eigenenergy is E_2 will become the new ground state (this can also be easily seen in the plot). hence, finding the eigenstate for this eigenenergy in the new hamiltonian, we get $|E_2\rangle = \begin{pmatrix} \frac{i(-3+\sqrt{9+h^2})}{h} \\ 1 \\ 0 \end{pmatrix}$ or, if we take h to infinity, we shall get: $|E_2\rangle = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$. if we represent this vector in the linear basis, we shall get a state proportional to $|\Downarrow\rangle$. and so the spin of the particle must be $m = -1$