E614: Symmetry breaking perturbation,degeneracy removing

Submitted by: Chapsky Alexandra

The problem:

Hamiltonian of a particle with spin 2 is

$\mathcal{H} = 2J_3^2 + J_1^2 + J_2^2 + 2\lambda(J_1^2 - J_2^2)$

We can easily see that the hamiltonian is diagonal in the standard base for $\lambda = 0$.

(1) Find the eigenstates of the Hamiltonian without the perturbation.
(2) Sketch the diagram of the energy levels and show which states the perturbation is coupling.
(3) Find the eigenvalues and eigenstates of the first excited level.
(4) Find up to second order the change of the lowest level due to the parturbation.

The solution:

(1)

$J^2 = J_1^2 + J_2^2 + J_3^2$

So we can write the hamiltonian without the perturbation :

$H_0 = J^2 + J_3^2$

And the eigenstates are:

$H_0 \left|\psi\right\rangle = (j(j+1) + m^2) \left|\psi\right\rangle$

For $j = 2 : m = 0, \pm1, \pm2 \Rightarrow E = 6, 7, 7, 10, 10$

$$\mathrm{H}_0 = \begin{pmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$

Degeneracy appears due to m^2, for $m \neq 0$.

(2)As we can see on the 614fig.bmp (external file), the perturbation couples the degenerated levels $E = 10$ and $E = 7$.

(3) We define $J_{\pm} = J_1 \pm iJ_2 \Rightarrow$

$\mathcal{H} = H_0 + \lambda V$

when $V = \lambda(J_+^2 + J_-^2)$

We arrange or basis in the following way:

$|2\rangle \, |-2\rangle \, |1\rangle \, |-1\rangle \, |0\rangle$

Since $J \pm |m\rangle = \sqrt{j(j+1) - m(m \pm 1)} \, |m \pm 1\rangle$ we get

$$\mathrm{V} = \begin{pmatrix} 0 & 0 & 0 & 0 & 2\sqrt{6} \\ 0 & 0 & 0 & 0 & 2\sqrt{6} \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 2\sqrt{6} & 2\sqrt{6} & 0 & 0 & 0 \end{pmatrix}$$

For the first excited state $m = \pm 1$. We see that for the block $|1\rangle \, |-1\rangle$ we get the following subspace:

$$H_{m=\pm 1} = \begin{pmatrix} 7 & 6\lambda \\ 6\lambda & 7 \end{pmatrix}$$

The eigen energies are:

$E_+ = 7 + 6\lambda$ and

$E_- = 7 - 6\lambda$

and the eigen vectors are :

$|+\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |-1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |-1\rangle)$

(4)

The first order correction is zero because:

$V_{[0,0]} = 0$

For the degenerated levels:

$$E_0^{[2]} = \sum_{m \neq 0} \frac{|V_{m,0}|^2}{E_0 - E_m} = \frac{(2\sqrt{6})^2}{6 - 10} + \frac{(2\sqrt{6})^2}{6 - 10} = -12$$

Hence, the lowest level, with $E^{[0]} = 6$ up to second order is:

$E_0 = 6 - 12\lambda^2$