E612: Harmonic oscillator with perturbation

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The problem:

Adding to the Hamiltonian of a harmonic oscillator with frequency ω a pertubation of the form $\hat{H}_1 = \lambda\hat{x}$.

(1) Find the energy of the ground state up to the second order using the perturbation theory and by exact calculation. compare the tow results.

(2) For perturbation of the form $\hat{H}_1 = \lambda\hat{x}^4$.
Find the first order correction to the n'th energy level.

The solution:

(1) The new Hamiltonian of the system will be:

$$\hat{H} = \hat{H}_0 + \lambda\hat{x} \quad where \quad \hat{H}_0 \mid n^{(0)}\rangle = \hbar\omega(n + 1/2) \mid n^{(0)}\rangle$$

$\mid n^{(0)}\rangle$ are the unperturbated eigenstates of Harmonic Oscillator.

The energy of the ground state expended to a series will be:

$$E_0 = \sum \lambda^k E_0^{(k)} = E_0^{(0)} + \lambda E_0^{(1)} + \lambda^2 E_0^{(2)} + O(\lambda^2)$$

Using the perturbation theory formulas (there is no degenerecy in harmonic oscillator so we can use them) :

$$E_n^{(1)} = \langle n^{(0)} \mid \hat{V} \mid n^{(0)}\rangle \tag{1}$$

$$E_n^{(2)} = \sum_{m(\neq n)} \frac{|\langle m^{(0)} \mid \hat{V} \mid n^{(0)}\rangle|^2}{E_n^{(0)} - E_m^{(0)}} \tag{2}$$

$$where \quad \hat{V} = \hat{x}$$

and by using the *lowering* and *raising* operators $\hat{a}$ and $\hat{a}^\dagger$ and their properties:

$$\hat{x} = \sqrt{\frac{\hbar}{2M\omega}}(\hat{a} + \hat{a}^\dagger) \quad ; \quad \mid n^{(0)}\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} \mid 0^{(0)}\rangle$$

As we expect the mean value of $\hat{x}$ at the ground state will be zero therefore:

$$E_0^{(1)} = \langle 0^{(0)} \mid \hat{x} \mid 0^{(0)}\rangle = 0$$

To find $E_0^{(2)}$ we will first calculate the terms :

$$x_{m0} = \langle m^{(0)} \mid \hat{x} \mid 0^{(0)}\rangle$$

After some algebra one can see that :

$$\hat{x} \mid 0^{(0)}\rangle = \sqrt{\frac{\hbar}{2M\omega}}(\hat{a} + \hat{a}^\dagger) \mid 0^{(0)}\rangle = \sqrt{\frac{\hbar}{2M\omega}} \mid 1^{(0)}\rangle$$

Therefore:

$$x_{m0} = \frac{1}{\sqrt{m!}}\sqrt{\frac{\hbar}{2M\omega}}\langle 0^{(0)} \mid \hat{a}^m \mid 1^{(0)}\rangle = \sqrt{\frac{\hbar}{2M\omega}}\delta_{m1}$$

Substituting into (2) we get:

$$E_0^{(2)} = -\frac{1}{2M\omega^2}$$

Therefore the enrergy of the ground state up to second order in λ will be:

$$E_0 = \frac{1}{2}\hbar\omega - \frac{\lambda^2}{2M\omega^2}$$

We can see that the ground state shifts down when this perturbation is present.
The other conclution is that this perturbation couples only the nearest neighbour states.

By exact calculation we will have to find the eigenvalues of the Hamiltonian:

$$\hat{H} = \frac{\hat{p}^2}{2M} + \frac{1}{2}M\omega^2\hat{x}^2 + \lambda\hat{x}$$

Changing variables to : $\hat{x} = \hat{y} + \frac{\lambda}{M\omega^2}$ yields the Hamiltonian of translated harmonic oscillator with the eigenvalue of the ground state :

$$E_0 = \frac{1}{2}\hbar\omega - \frac{\lambda^2}{2M\omega^2}$$

Therefore the result that we have got by the perturbation theory is exact, which means that higher order corrections to the energy are all zero.

(2) Using (1) for the perturbation of the form $\hat{H}_1 = \lambda\hat{x}^4$ we get :

$$E_n^{(1)} = \langle n^{(0)} \mid (\sqrt{\frac{\hbar}{2m\omega}})^4(\hat{a} + \hat{a}^\dagger)^4 \mid n^{(0)}\rangle$$

The only terms that are non-zero are:

$$\langle n^{(0)} \mid \hat{a}^2\hat{a}^{\dagger 2} + \hat{a}^{\dagger 2}\hat{a}^2 + 4\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a} + 4\hat{a}^\dagger\hat{a} + 1 \mid n^0\rangle = 6n^2 + 6n + 3$$

Therefore we get that the first order correction to the n'th energy level will be:

$$E_n^{(1)} = (\frac{\hbar}{2m\omega})^2\lambda(6n^2 + 6n + 3)$$

We see that this perturbation couples to the n'th state the states (first order) :

$$\mid (n-2)^{(0)}\rangle \ ; \ \mid (n+2)^{(0)}\rangle \ ; \ \mid (n-1)^{(0)}\rangle \ ; \ and \mid (n+1)^{(0)}\rangle$$