Ex608: A partical with spin in a box + scaterring

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The problem:

A partical with spin 1/2 and mass M is placed in a one dimensional box. Assume boundary condition to be zero at the edges of the interval $-L/2 < x < L/2$. The Hamiltonian of the system is $\mathcal{H} = \frac{p^2}{2M} + \epsilon\delta(x)(\frac{1}{2} + S_z)$.

(1) Write the eigenvalues and the eigenstates (in the standard represantation) while $\epsilon = 0$.
(2) Calculate the first order perturbation for the energies for a small ϵ. Is there degeneration left in the spectrum?
(3) Write the energies of the partical when $\epsilon \to \infty$. Explain the degeneration of the spectrum.
(4) Draw the energies of the first 8 lowest states as a function of ϵ.

The solution:

(1) For $\epsilon = 0$ the Hamiltonian is $\mathcal{H} = \frac{p^2}{2M}$. The spin of the partical is $m = \pm\frac{1}{2}$.

The eigenstates are:

$$|n, even\rangle = \sqrt{\frac{2}{L}}\cos(k_n x) \otimes |\pm\rangle$$

$$|n, odd\rangle = \sqrt{\frac{2}{L}}\sin(k_n x) \otimes |\pm\rangle$$

We define: $k_n = \frac{\pi}{L}n$

Then the energies are:

$$E_n = \frac{k^2}{2M} = \frac{\pi^2}{2ML^2}n^2$$

Where:

$n = 1, 3, 5...$ for the even states.
$n = 2, 4, 6...$ for the odd states.

(2) The first order perturbation for the energies for a small ϵ is:

$$E^{[1]} = V_{n_o,n_o} = \left\langle \psi^{n_o} \middle| \epsilon\delta(x)(\frac{1}{2} + S_z) \middle| \psi^{n_o} \right\rangle$$

S_z is constant of motion and therefore the Hamiltonain separates into:

$$\mathcal{H}^{(+)} = \frac{p^2}{2M} + \epsilon\delta(x)$$

$$\mathcal{H}^{(-)} = \frac{p^2}{2M} + 0$$

Now we can easely say that:
The m=-1/2 (“down” spin) states are not affected by the pertubration.

Moreover, since the odd states vanish at the origin they are also not affected by the perturbation. So the only states that are affected by the perturbation are even states with "up" spin.

$$E^{[1]}_{down} = 0$$

$$E^{[1]}_{odd} = 0$$

$$E^{[1]}_{even,up} = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \cos^2(k_n x)\epsilon\delta(x)dx = \frac{2}{L}\epsilon$$

So, we get:

$$E_{even,up} = E^{[0]}_{even} + \frac{2}{L}\epsilon$$

There still is degeneration left in the spectrum due to the n_{odd} states.

(3) For $\epsilon \longrightarrow \infty$ we have another boundary condition:

$\psi_{(x)} = 0$ at $x = 0$

Now we have a new problem with a partical that can be in one of two places: $-L/2 < x < 0$ or $0 < x < L/2$, with a finite probability to be in each side.
The n_{odd} states degenerated with the n_{even} states.

The new eigenstates are:

$$|R\rangle = \theta(x)\sin(k'_n x)$$

$$|L\rangle = \theta(-x)\sin(k'_n x)$$

Where:

$$\theta(x) = \begin{cases} 1, x > 0 \\ 0, x < 0 \end{cases}$$

$$k'_n = \frac{\pi}{L/2}n$$

The energies are:

$$E_n = \frac{k'^{\,2}_n}{2M} = \frac{\pi^2}{2M(L/2)^2}n^2$$

Where: $n = 1, 2, 3...$

Now we have degeneration in the spectrum due to the spin and due to n_{even}/n_{odd} states.