E6070: Perturbation Theory for a ring in an Electric Field

Submitted by: Jacob Hanouna and Sarit Nagar

The problem:

A particle with mass M and charge e is placed in a one dimensional ring with radius R.
A homogenous electric field $\mathcal{E}$ is being induced parallel to the ring's plane.

(1) Write the Hamiltonian $\mathcal{H}(\theta, p)$ of the system.
(2) What are the symmetries the system has without/with the perturbation.
(3) Write the matrical presentation of the Hamiltonian with the suitable basis for the perturbated system.
(4) Calculate the ground state energy $n = 0$ up to the second order in the perturbation.
(5) Calculate the excited states energies $n > 1$ up to the second order in the perturbation.

The solution:

(1) The Hamiltonian is:

$$\hat{\mathcal{H}} = \frac{\hat{p}^2}{2m} + \hat{V}(\theta) = \frac{\hat{p}^2}{2m} + e\mathcal{E}R\cos\theta$$

(2) Without the perturbation the Hamiltonian has symmetry with respect to rotations and reflections. In the presence of the electric field, only the reflection remains.

(3) We will use the basis that complies with the reflection symmetry:

$$|n=0\rangle \mapsto \frac{1}{\sqrt{2\pi R}} \longmapsto \frac{1}{\sqrt{2\pi}}$$

$$|n,+\rangle \mapsto \frac{1}{\sqrt{\pi R}}\cos(k_n x) \longmapsto \frac{1}{\sqrt{\pi}}\cos(n\theta)$$

$$|n,-\rangle \mapsto \frac{1}{\sqrt{\pi R}}\sin(k_n x) \longmapsto \frac{1}{\sqrt{\pi}}\sin(n\theta)$$

Notice that describing the basis using the relation $x = R\theta$ changes the normalization as mentioned on the right side.
There is a degeneracy between the even states and the odd states, though they are not "coupled" to each other. Therefore, the Hamiltonian will be of the form:

$$\mathcal{H} \mapsto \begin{pmatrix} \varepsilon_0 & & & & & & \\ & \varepsilon_1 & & & & & \\ & & \varepsilon_2 & & & & \\ & & & \ddots & & & \\ & & & & \varepsilon_1 & & \\ & & & & & \varepsilon_2 & \\ & & & & & & \ddots \end{pmatrix} + \begin{pmatrix} V_{00} & V_{01}^{(+)} & V_{02}^{(+)} & \cdots & V_{01}^{(-)} & V_{02}^{(-)} & \cdots \\ V_{10}^{(+)} & V_{11}^{(+)} & V_{12}^{(+)} & \cdots & 0 & 0 & \cdots \\ V_{20}^{(+)} & V_{21}^{(+)} & V_{22}^{(+)} & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ V_{10}^{(-)} & 0 & 0 & \cdots & V_{11}^{(-)} & V_{12}^{(-)} & \cdots \\ V_{20}^{(-)} & 0 & 0 & \cdots & V_{21}^{(-)} & V_{22}^{(-)} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $\varepsilon_n = \frac{k_n^2}{2M} = \frac{n^2}{2MR^2}$.

$$V_{00} = \langle n=0|V(\theta)|n=0\rangle = 0$$

$$V^{(+)}_{0m} = \langle n=0|V(\theta)|m,+\rangle = \frac{e\mathcal{E}R}{\sqrt{2}\pi}\int_0^{2\pi}\cos(\theta)\cos(m\theta)d\theta = \frac{e\mathcal{E}R}{\sqrt{2}}\delta_{m,1}$$

$$\begin{aligned}
V^{(+)}_{nm} &= \langle n,+|V(\theta)|m,+\rangle = \frac{e\mathcal{E}R}{\pi}\int_0^{2\pi}\cos(n\theta)\cos(\theta)\cos(m\theta)d\theta \\
&= \frac{e\mathcal{E}R}{4\pi}\left[\int_0^{2\pi}\left[\cos((n-m-1)\theta)+\cos((n-m+1)\theta)\right]d\theta \quad + \right. \\
&\quad \left. + \int_0^{2\pi}\left[\cos((n+m-1)\theta)+\cos((n+m+1)\theta)\right]d\theta \right. \\
&= \frac{e\mathcal{E}R}{2}\left[\delta_{n,m+1}+\delta_{n,m-1}+\delta_{n,-m+1}+\delta_{n,-m-1}\right] = \frac{e\mathcal{E}R}{2}\left[\delta_{n,m+1}+\delta_{n,m-1}\right]
\end{aligned}$$

The last transition is due to the fact that $n, m > 0$.

$$V^{(-)}_{0m} = \langle n=0|V(\theta)|m,-\rangle = \frac{e\mathcal{E}R}{\sqrt{2}\pi}\int_0^{2\pi}\cos(\theta)\sin(m\theta)d\theta = 0$$

$$V^{(-)}_{nm} = \langle n,-|V(\theta)|m,-\rangle = \frac{e\mathcal{E}R}{\pi}\int_0^{2\pi}\sin(n\theta)\cos(\theta)\sin(m\theta)d\theta = \frac{e\mathcal{E}R}{2}\left[\delta_{n,m+1}+\delta_{n,m-1}\right]$$

$$\mathrm{V}^{(+)}_{n,m} = \tfrac{e\mathcal{E}R}{2}\begin{pmatrix} 0 & \sqrt{2} & 0 & 0 & \cdots \\ \sqrt{2} & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad V^{(-)}_{n,m} = \tfrac{e\mathcal{E}R}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

(4) Notice that $E^{[0]}_n = \varepsilon_n$ and $E^{[1]}_n = V_{nn} = 0$ Therefore, the ground state energy up to the second order in the perturbation is simply

$$E_0 = E^{[2]}_0 = \sum_{m\neq 0}\frac{|V_{0m}|^2}{0-\frac{m^2}{2MR^2}} = -2Me^2\mathcal{E}^2R^4$$

(5) For the excited states $n, m > 1$ we have $V^{(+)}_{n,m} = V^{(-)}_{n,m}$

$$E^{[2]}_n = \sum_{m\neq n}\frac{|V_{nm}|^2}{\varepsilon_n-\varepsilon_m} = \frac{Me^2\mathcal{E}^2R^4}{2}\left[\frac{1}{n^2-(n-1)^2}+\frac{1}{n^2-(n+1)^2}\right] = \frac{Me^2\mathcal{E}^2R^4}{4n^2-1}$$

Therefore the energy up to second correction is:

$$E_n = \frac{n^2}{2MR^2} + \frac{Me^2\mathcal{E}^2R^4}{4n^2-1}$$