E607: Perturbation theory for a ring + scatterer

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The problem:
Consider a particle with mass m and charge q, on a one dimensional ring, length L.
There is a scattere in the ring which described by $V(x) = u\cos(\frac{2\pi x}{L})$.
Calculate the second order correction to the energy in u (for any n).

The Solution:

`Should be polished.`

The Hamiltonian that describes the system is:

$$H = \frac{p^2}{2m} + V(x)$$

The Hamiltonian has symmetry with respect to reflections. The basis we chose to work with complies with the reflection symmetry (the even/odd states):

$$|n=0>=\sqrt{\frac{1}{L}}$$

$$|n_{odd}>=\sqrt{\frac{2}{L}}\sin(\frac{2\pi nx}{L})$$

$$|n_{even}>=\sqrt{\frac{2}{L}}\cos(\frac{2\pi nx}{L})$$

The perturbation does not couple the odd states to the even states. So we focus on the perturbation's influence on the even states, and the odd states.

$$V_{nm} =< m|V(x)|n>$$

For the even states:

$$V_{0,1} = V_{1,0} = \int_L^0 \frac{2}{L}\cos(\frac{2\pi x}{L})u\cos(\frac{2\pi x}{L})dx = \frac{u}{\sqrt{2}}$$

$$V_{nm}(even) = \int_L^0 \frac{2}{L}\cos(\frac{2\pi nx}{L})\cos(\frac{2\pi mx}{L})u\cos(\frac{2\pi x}{L})dx$$

$$V_{nm}(even) = \frac{2}{L}u\int_L^0 \frac{1}{2}(\cos(\frac{2\pi x(n-m)}{L} + \cos(\frac{2\pi x(n+m)}{L}))\cos(\frac{2\pi x}{L})dx = \frac{u}{2}\delta_{|n-m|,1}$$

For the odd states:

$$V_{nm}(odd) = \int_L^0 \frac{2}{L}\sin(\frac{2\pi nx}{L})\sin(\frac{2\pi mx}{L})u\cos(\frac{2\pi x}{L})dx$$

$$V_{nm}(odd) = \frac{2}{L}u\int_L^0 \frac{1}{2}(\cos(\frac{2\pi x(n-m)}{L} - \cos(\frac{2\pi x(n+m)}{L}))\cos(\frac{2\pi x}{L})dx = \frac{u}{2}\delta_{|n-m|,1}$$

The perturbation matrix for the block that belongs to the even and odd states:

$$V_{nm}(even) = \frac{u}{2}\begin{pmatrix} 0 & \sqrt{2} & 0 & ...... \\ \sqrt{2} & 0 & 1 & ...... \\ 0 & 1 & 0 & ...... \\ 0 & 0 & 1 & \\ . & & & . \\ . & & & \end{pmatrix}$$

$$V_{nm}(odd) = \frac{u}{2}\begin{pmatrix} 0 & 1 & 0 & .... & \\ 1 & 0 & 1 & .... & \\ 0 & 1 & 0 & .... & \\ 0 & 0 & 1 & & \\ . & & & . & \\ . & & & & . \end{pmatrix}$$

Therefore the corrections to the energy:
The first order corrections to the energy levels:

$$E^{[1]}_{n\neq 0} = V_{n_0,n_0} = 0$$

The zero order energy is given by:

$$E_n = \frac{2\pi^2 n^2}{mL^2}$$

So the correction of the second order is:
For $n = 0$:

$$E^{[2]}_0 = \sum_{l\neq 0} \frac{|<0|V|l>|^2}{E_0 - E_l} = \sum_{l\neq 0} \frac{|V_{ol}|^2}{0 - \frac{2\pi^2 l^2}{mL^2}} = -\frac{mL^2}{4\pi^2}u^2$$

For $n = 1, 2, 3......$:

$$E^{[2]}_n = \sum_{l\neq n} \frac{|<n|V|l>|^2}{E_n - E_l} = \sum_{l\neq 0} \frac{|V_{nl}|^2}{\frac{2\pi^2 n^2}{mL^2} - \frac{2\pi^2 l^2}{mL^2}} = \sum_{l\neq n} \frac{1}{\frac{2\pi^2 n^2}{mL^2} - \frac{2\pi^2 l^2}{mL^2}}(\frac{u}{2})^2$$

The matrix elements are all zero,except for l=n+1 and l=n-1.So the expression for the second order correction is:

$$E^{[2]}_n = \frac{mL^2}{8\pi^2}(\frac{1}{2n-1} - \frac{1}{2n+1})u^2$$

And the correction to the ground state energy, up to the second order is:

$$E_n = E^{[0]}_n + E^{[1]}_n + E^{[2]}_n$$

For $n = 0$:

$$E_0 = 0 + 0 - \frac{mL^2}{4\pi^2}u^2 = -\frac{mL^2}{4\pi^2}u^2$$

For $n \neq 0$:

$$E_n = \frac{2n^2\pi^2}{mL^2} + 0 + \frac{mL^2}{8\pi^2}(\frac{1}{2n-1} - \frac{1}{2n+1})u^2$$

$$E_n = \frac{2n^2\pi^2}{mL^2} + \frac{mL^2}{8\pi^2}(\frac{1}{2n-1} - \frac{1}{2n+1})u^2$$