Ex5668: Spin precession due to spin-orbit interaction

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The problem:

An electron, with mass M, charge e and gyromagnetic constant g, launched with energy E in a one dimensional conductor, in the direction of the X axis. The conductor passes through capacitor plates of length L. The capacitor creates an electric field $\mathcal{E}$ in the Y direction. Likewise, there's a magnetic field B in that area in the Z direction. When the electron enters the region of interaction, its spin is being polarized in the direction of its motion.

Throughout the problem we may ignore the possibility of the particle returning from the interaction region. And also $\hbar = 1$.

In the first two paragraphs $v \approx (2E/M)^{1/2}$

(1) What will be the direction ϕ of the spin, when the electron escapes the interaction region?
(2) What does the magnetic field has to be, for the spin not to rotate?

In the next paragraphs $x = 0$ will be set as the entering point. Also giving an exact solution on the basis of the launching energy.

(3) Write the particle's state in the region where the interaction takes place, in the standard basis $|x, m\rangle$, where $m = \uparrow, \downarrow$.
(4) Give the exact answer to paragraph (1).

The solution:

(1) The Hamiltonian of the motion is as follows:

$$\mathcal{H} = \frac{p^2}{2M} - g\frac{e}{2M}\mathbf{B}\cdot\mathbf{S} - \frac{e}{2M^2}(\mathcal{E}\times\mathbf{p})\cdot\mathbf{S}$$

$$\mathcal{H} = \frac{p^2}{2M} - g\frac{eB_z}{2M}S_z + \frac{ev\mathcal{E}_y}{2M}S_z = \frac{p^2}{2M} - \frac{eS_z}{2M}(B_z g - v\mathcal{E}_y)$$

The time that it takes the electron, to get out of the capacitor is:

$$t = \frac{L}{v} \approx L\left(\frac{M}{2E}\right)^{\frac{1}{2}}$$

We also know that since S_z is the rotations generator around the z direction, it has to satisfy, (up to a phase):

$$R(\bar{e}_z\Phi) = e^{-i\Phi S_z} = e^{-i\mathcal{H}t}$$

Inserting the Hamiltonain explicitly:

$$e^{-i\mathcal{H}t} = e^{-i\frac{p^2}{2M}\cdot t}\cdot e^{i\frac{eS_z}{2M}(B_z g - v\mathcal{E}_y)\cdot t}$$

Now by comparing the angles related to S_z, and with respect to time t ,The angle at the end of the interaction zone is:

$$\Phi = -\frac{e}{2M}(B_z g - v\mathcal{E}_y)\times\frac{L}{v}$$

(2) For the spin not to change, we must have $\Phi = 0$

$$\Phi = -\frac{e}{2M}(B_z g - v\mathcal{E}_y) \times \frac{L}{v} = 0$$

And therefore:

$$B_z = \frac{v\varepsilon}{g}$$

(3) Considering the degrees of freedom in the problem, we could write the general wavefunction as:

$$|\psi\rangle = \sum \psi_{x,m}|x, m\rangle$$

Where

$$|x, m\rangle = |x\rangle \otimes |m\rangle \text{ and } m = \uparrow, \downarrow$$

So we get:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(e^{ik_\uparrow x} \otimes |\uparrow\rangle + e^{ik_\downarrow x} \otimes |\downarrow\rangle\right)$$

Since the possible values for m are: $m = \pm\frac{1}{2}$

$$\Rightarrow k_m = \sqrt{2ME \pm \frac{1}{2}e(B_z g - v\mathcal{E}_y)}$$

For $m = \frac{1}{2}$ we write:

$$k_\uparrow = \sqrt{2ME + \frac{1}{2}e(B_z g - v\mathcal{E}_y)}$$

And for $m = -\frac{1}{2}$ we write:

$$k_\downarrow = \sqrt{2ME - \frac{1}{2}e(B_z g - v\mathcal{E}_y)}$$

(4) Looking at the definition of $|\psi\rangle$, we can easily see that for $x = 0$ we get:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

And at $x = L$:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{ik_\uparrow L}\\e^{ik_\downarrow L}\end{pmatrix}$$

We can write the following equation:

$$e^{-i\Phi S_z}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}e^{-i\frac{\Phi}{2}} & 0\\0 & e^{i\frac{\Phi}{2}}\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}e^{ik_\uparrow L}\\e^{ik_\downarrow L}\end{pmatrix}$$

By solving the above equation:

$$\Phi = -(k_\uparrow - k_\downarrow) \cdot L$$

The above solution is accurqate up to a global phase of: $e^{i\left(k_\uparrow + k_\downarrow\right)\cdot\frac{L}{2}}$.