E5664: Spin-orbit interaction

Submitted by: Snir Cohen

The problem:

An electron of mass M and charge e is located in a one-dimensional ring of radius R. The electron has a spin-orbit interaction with the electric field that creates a charge Q that is located in the center of the ring. It is recommended to use units of $\hbar$ =c= 1.

(1) Write the Hamiltonian $H(x, p, S)$ of the electron.
(2) What are the eigenenergies of the electron?
(3) What is the velocity of the electron in each one of the energy states?
(4) What is the magnetic flux Φ through the center of the ring that will eliminate the effect of Q on an electron that has spin up?
Express all the answers by the given data alone.

The solution:

(1) The general Hamiltonian of a spin-orbit interaction:

$$\mathcal{H} = \frac{p^2}{2M} + \mathcal{H}_{SO}$$

The spin-orbit term:

$$\mathcal{H}_{SO} = -\frac{e}{2M^2}(\mathcal{E} \times \vec{p}) \cdot \vec{S} = -\frac{eQ}{2M^2R^2}pS_z$$

Where the electric field

$$\mathcal{E} = \frac{Q}{R^2}$$

Hence

$$\mathcal{H} = \frac{1}{2M}\left(p^2 - \frac{eQ}{MR^2}pS_z\right)$$

We complete the square and get

$$\mathcal{H} = \frac{1}{2M}\left(p - \frac{eQ}{2MR^2}S_z\right)^2 - \frac{1}{2M}\left(\frac{e^2Q^2S_z^2}{4M^2R^4}\right)$$

(2) From periodic boundary conditions:

$$k = \frac{2\pi n}{L} = \frac{n}{R} \quad , \quad n = \text{integer}$$

Where

$$L = 2\pi R$$

For an electron

$$m = \pm\frac{1}{2}$$

Hence the eigenenergies are:

$$E_{n,\pm} = \frac{1}{2M}\left(\frac{n}{R} \mp \frac{eQ}{4MR^2}\right)^2 - \frac{e^2Q^2}{32M^3R^4}$$

(3) The definition of the velocity operator is implied by the "rate of change fomula":

$$\hat{v} = i[\mathcal{H}, \hat{x}] \;\;,\;\; \mathcal{H} = \frac{1}{2M}\left(p^2 - \frac{eQ}{MR^2}pS_z\right)$$

$$\hat{v} = i \cdot \frac{1}{2M}\left([\hat{p}^2, \hat{x}] - \frac{eQ}{MR^2}\hat{S}_z[\hat{p}\;, \hat{x}]\right) = i \cdot \frac{1}{2M}\left(-2i\hat{p} + i \cdot \frac{eQ}{MR^2}\hat{S}_z\right)$$

$$\Rightarrow \hat{v} = \frac{1}{M}\left(\hat{p} - \frac{eQ}{2MR^2}\hat{S}_z\right)$$

Hence the velocities are:

$$v_{n,\pm} = \frac{1}{M}\left(\frac{n}{R} \mp \frac{eQ}{4MR^2}\right)$$

(4) Spin up:

$$m = \frac{1}{2}$$

The effect of a magnetic flux through the center of the ring:

$$E_{n,+} = \frac{1}{2M}\left(\frac{n}{R} - \frac{eQ}{4MR^2} - \frac{e\Phi}{L}\right)^2 + const$$

$$\frac{e\Phi}{L} = \frac{e\Phi}{2\pi R}$$

We demand

$$-\frac{eQ}{4MR^2} - \frac{e\Phi}{2\pi R} = 0$$

So

$$\Phi = -\frac{\pi Q}{2MR}$$