E6010: The spin-orbit interaction

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The problem:

An electron with mass M and charge e is free to move on a single dimensional ring of radius R. The electron has a spin-orbit interaction with the electric field created by a charge Q located in the center of the ring. It's recommended to use the units $\hbar$=c=1

(1) Write the Hamiltonian $H(x, p, s)$.
(2) Find the eigenenergies of the electron.
(3) Find the speed of the electron for each energy state.
(4) What magnetic flux (Φ) through the ring would negate the effect the charge Q has over en electron with spin up?

Give the answer in terms of the given parameters only.

The solution:

(1) The Hamiltonian that includes the spin-orbit interaction is as following

$$(a)\mathcal{H} = \frac{1}{2m}p^2 - \frac{e}{2m^2}(\vec{\epsilon} \times \vec{p}) \cdot \vec{s}$$

(2) The electtric field created by a point charge is given by:

$$(b)\vec{\epsilon} = \frac{Q}{R^2}\hat{r}$$

We'll write the momentum as:

$$(c)\vec{p} = p\hat{\phi}$$

The spin of a electron is:

$$(d)\vec{s} = s\hat{z}$$

Substituting (d), (c) and (b) into (a) we get:

$$(e)\mathcal{H} = \frac{1}{2m}p^2 - \frac{Qes}{2m^2R^2}p$$

Notice that the Hamiltonian can be rewritten as:

$$\mathcal{H} = \frac{1}{2m}\left(p - \frac{Qes}{2mR^2}\right)^2 - \frac{Q^2e^2s^2}{8m^3R^4}$$

As seen in the Aharonov-Bohm effect, if we have shifted momentum, the eigentsates will be as those as of a regular momentum. Addition of a constant won't affect the eigenstates. The eigenvalues of a Hamilitonian with momentum shifted by $+a$ will be as following:

$$(f)E_n = \frac{1}{2m}\left(\frac{2\pi\hbar}{L}n + a\right)^2$$

The constant addition to the Hamiltonian will appear in eigenvalues as a constant addition. Substituting (e) into (f) and adding the contant we get:

$$(g)E_n = \frac{1}{2m}\left(\frac{2\pi\hbar}{L}n + \frac{Qes}{2mR^2}\right)^2 - \frac{Q^2e^2s^2}{8m^3R^4}$$

Finally, substituting L with $2\pi R$ and opening the parentheses we end up with:

$$E_n = \frac{1}{2mR^2}\left(n^2 + \frac{Qes}{2mR}\right)$$

(3) Because the eigenstates of the energies are the eigenstates of momentum:

$$v = \frac{1}{mR}\left(n + \frac{Qes}{2mR}\right)$$

(4) We know that flux Φ will appear in the Hamiltonian as shift in momentum. All we need is for two shifts to cancel each other:

$$\frac{Qes}{2mR^2} - \frac{e\Phi}{2R\pi} = 0$$

$$\Phi = \frac{\pi sQ}{mR}$$