Ex5662: Landau levels in Graphene

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The problem:

The effective Hamiltonian of an electron in a 2-D layer of Graphene is $H = v_0\boldsymbol{\sigma}(\boldsymbol{p} - e\boldsymbol{A})$ while $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, 0)$ and $\boldsymbol{A}$ is the vector potential for a perpendicular magnetic field $\boldsymbol{B}$ within the Landau gauge.
Notice that the standard basis for an electron is $|x, y, m\rangle$, $m =\downarrow, \uparrow$.

(1)In the absence of a magnetic field, for a given momentum $\boldsymbol{p} = (p_x, p_y)$, what are the eigenenergies of the particle?
(2)Within the Landau gauge $\hat{Y} = -(\frac{1}{eB})\hat{p_x}$ is a constant of motion. Write down the Hamiltonian $H^{(Y)} = H_{m,m'}(p_y, y-Y)$ after the variable separation. Note that the linear combination of canonical coordinates $a = \frac{(sQ+\frac{iP}{s})}{\sqrt{2}}$ is a ladder operator, and write down the Hamiltonian derived in the alternative notation $H_{m,m'}(a, a^\dagger)$
(3)Define the operator $C = (H^{(Y)})^2$.
Calculate the eigenvalues $\lambda_{n=0,1,2...}$ of C,
notice, the eigenvalues are degenerated except $n = 0$.
(4)Since C is a constant of motion it is possible to use variable separation once more.
Write down the $2X2$ matrix representing $H^{(Y,n)}$.
(5)What are the energy levels $E_{Y,n,\pm}$ for $n > 0$?
(6)Find the eigenstates and write them in the standard basis.
Notice that the quantum state in the standard basis in represented by $\Psi \mapsto (\psi_\uparrow(x, y), \psi_\downarrow(x, y))$.
You may use the notation $\varphi^n()$ for the eigenfunctions of a 1-D harmonic oscillator.

The solution:

(1)The Hamiltonian without a magnetic field is:

$$H = v_0\boldsymbol{\sigma} \cdot \boldsymbol{p} = v_0(\sigma_x p_x + \sigma_y p_y)$$

$$= v_0 \begin{pmatrix} 0 & p_x - ip_y \\ p_x + ip_y & 0 \end{pmatrix} = v_0|\boldsymbol{p}| \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}$$

where $\boldsymbol{p} = (p_x, p_y)$ is given and $tg(\phi) = \frac{p_y}{p_x}$.
Diagonalization of this Hamiltonian provides the following eigenenergies: $E_\pm = \pm v_0|\boldsymbol{p}|$.

(2)The vector potential within the Landau gauge is:

$$\boldsymbol{A} = B(-y, 0, 0)$$

Therefore the Hamiltonian is:

$$H = v_0[\sigma_x(p_x + eB_y) + \sigma_y p_y]$$

plugging the definition of Y into H yields:

$$H = v_0[eB\sigma_x(y - Y) + \sigma_y p_y]$$

One should notice that $[Y, H] = 0$, thus the Hamiltonian is separable into blocks of the form:

$$H^{(Y)} = v_0[eB\sigma_x(y - Y) + \sigma_y p_y] = v_0\sqrt{eB}[\sqrt{eB}\sigma_x(y - Y) + \frac{1}{\sqrt{eB}}\sigma_y p_y]$$

We define:

$$Q = y - Y, P = p_y, s = \sqrt{eB}$$

So:

$$H^{(Y)}(Q,P) = v_0[\sigma_x\sqrt{eB}Q + \frac{1}{\sqrt{eB}}\sigma_y P]$$

Using the ladder operator given:

$$a = \frac{1}{\sqrt{2}}(Q\sqrt{eB} + \frac{i}{\sqrt{eB}}P)$$

$$\Downarrow$$

$$Q = \frac{1}{\sqrt{2eB}}(a^\dagger + a), P = \frac{\sqrt{eB}}{i\sqrt{2}}(a - a^\dagger)$$

plugging into the Hamiltonian:

$$H^{(Y)}(a,a^\dagger) = v_0\sqrt{2eB}[a(\sigma_x - i\sigma_y) + a^\dagger(\sigma_x + i\sigma_y)]$$

$$H^{(Y)} \mapsto v_0\sqrt{2eB}\begin{pmatrix} 0 & a^\dagger \\ a & 0 \end{pmatrix}$$

(3)

$$[a, a^\dagger] = 1 \Rightarrow aa^\dagger = 1 + a^\dagger a$$

$$C = (H^{(Y)})^2 \mapsto 2eBv_0^2\begin{pmatrix} a^\dagger a & 0 \\ 0 & aa^\dagger \end{pmatrix} = 2eBv_0^2\begin{pmatrix} a^\dagger a & 0 \\ 0 & a^\dagger a + 1 \end{pmatrix}$$

The eigenstates of $a^\dagger a$ satisfy:(1D harmonic oscillator)

$$a^\dagger a\,|n\rangle = n\,|n\rangle\,, (n = 0, 1, 2\ldots)$$

The eigenvectors of C are: $\begin{pmatrix} |n\rangle \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ |n-1\rangle \end{pmatrix}$

$$C^{(n)} \mapsto 2eBv_0^2\begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix} \Rightarrow \lambda_n = 2eBv_0^2 n$$

(4) $[C, H^{(Y)}] = 0$, therefore $H^{(Y)}$ is separable into blocks.

$$H^{(Y)}\begin{pmatrix} |n\rangle \\ 0 \end{pmatrix} = v_0\sqrt{2eB}\begin{pmatrix} 0 \\ a\,|n\rangle \end{pmatrix} = v_0\sqrt{2eBn}\begin{pmatrix} 0 \\ |n-1\rangle \end{pmatrix}$$

$$H^{(Y)}\begin{pmatrix} 0 \\ |n-1\rangle \end{pmatrix} = v_0\sqrt{2eB}\begin{pmatrix} a^\dagger\,|n-1\rangle \\ 0 \end{pmatrix} = v_0\sqrt{2eBn}\begin{pmatrix} |n\rangle \\ 0 \end{pmatrix}$$

$$\Downarrow$$

$$H^{(Y,n)} \mapsto v_0\sqrt{2eB}\begin{pmatrix} 0 & \sqrt{n} \\ \sqrt{n} & 0 \end{pmatrix}$$

(5)

$$H^{(Y,n)} = v_0\sqrt{2eBn}\sigma_x \Rightarrow E_{Y,n,\pm} = \pm v_0\sqrt{2eBn}, (n > 0)$$

(6) The eigenstates of $H^{(Y)}$ are (up to a normalization factor):

$$\begin{pmatrix} |n\rangle \\ \pm |n-1\rangle \end{pmatrix}$$

the representation of these eigenstates in the standard basis is:

$$\begin{pmatrix} \varphi^n(y) \\ \pm\varphi^{n-1}(y) \end{pmatrix}$$

remembering that Y shares the same eigenstates with p_x yields the following eigenstates:

$$e^{-ieBYx} \begin{pmatrix} \varphi^n(y) \\ \pm\varphi^{n-1}(y) \end{pmatrix}$$