Ex5662: Landau Levels in Grafene

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The problem:

The effective Hemiltonian of an electron in a 2-D layer of Grafene is $H = v_0\sigma(p - eA)$ when $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ and A is the vector potential for a pependicular magnetic field B in Landau gauge. Be advised that the standard base for representing the electron is $\mid x, y, m >$, when $m = \downarrow, \uparrow$.

(1) In the lack of a magnetic field, and for a given momentum $p = (p_x, p_y)$ what are the eigen energies of the particle?
(2) In Landau gauge $\widehat{Y} = -(1/(eB))\widehat{p_x}$ is a constant of motion. Write the Hemiltonian $H^Y = H_{m,m'}(p_y, y - Y)$ of after the seperation of variables. Please note that a linear combination of canonical coordinates $a = (sQ + iP/s)/\sqrt{2}$ is a ladder operator, and write the Hemiltonian in the alternative way $H_{m,m'}(a, a^\dagger)$
(3) Define the operator $C = (H^Y)^2$ and calculate the eigenvalues $\lambda_n = 0, 1, 2...$of C. Please note that all the eiegenvalue except n=0 are degenerated.
(4) Since C is a constant of motion it is possible to perform a second seperation of variables.Write the 2x2 matrix representing $H^{Y,n}$.
(5) Find the energy levels $E_{Y,n,\pm}$ for $n > 0$
(6) Write the matrix form of the operator $\hat{I}(t)$ and represent it as a sum of Pauli Matrices. Find the eigenstates and write them in the standard base. Be advised that the quantum state in the standard base in represented by $\bar{\ominus} \rightarrow (\ominus_\uparrow(x, y), \Psi_\downarrow(x, y))$. It is possible to make use of $\varphi^n()$ for the eigen functions of a 1-D harmonic oscilator.

The solution:

Preface: $H = V_0\sigma \cdot (p - e\bar{A})$, $\bar{B} = B\hat{z} \Longrightarrow \{Lnadau - Gauge\} \Longrightarrow \bar{A} = (-By, 0, 0)$

$\sigma = (\sigma_x, \sigma_y)$, $p = (p_x, p_y) \Longrightarrow \sigma \cdot p = \sigma_x p_x + \sigma_y p_y$, $\sigma \cdot e\bar{A} = -eBy\sigma_x$

$$\Longrightarrow H = v_0[\sigma_x(p_x + eBy) + \sigma_y p_y) = v_0 \begin{pmatrix} 0 & (p_x + eBy - ip_y) \\ (p_x + eBy + ip_{y)} & 0 \end{pmatrix}$$

$$\Longrightarrow E = \pm v_0(p_x^2 + 2p_x eBy + (eBy)^2 + p_y^2)^{1/2} = \pm v_0(\bar{p}^2 + 2p_x eBy + (eBy)^2)^{1/2}$$

(1) for $B = 0 \Longrightarrow E = \pm v_0 \, |\bar{p}|$

(2) $.H = V_0\sigma(\bar{p} + eBy\hat{x})$, Let us define $\widehat{Y} = -\widehat{p_x}/(eB)$ so that $[H, Y] = 0$

$$\Longrightarrow H = v_0[\sigma_x eB(-Y + y) + \sigma_y p_y) = v_0 eB \begin{pmatrix} 0 & y - Y - ip_y/eB \\ y - Y + ip_y/eB & 0 \end{pmatrix}$$

Let us define Q and P as canonical conjugates in the following manner:
$Q = y - Y$, $P = p_y$
and so we can define ladder oppeators a and $a^\dagger$in the following way:
$a = (sQ + iP/s)/\sqrt{2}$
$a^\dagger = (sQ - iP/s)/\sqrt{2}$
when $s = \sqrt{2eB}$ and the Hemiltonian will take the form:

$$H = v_0\sqrt{2eB} \begin{pmatrix} 0 & a^\dagger \\ a & 0 \end{pmatrix}$$

(3) $[a, a^\dagger] = aa^\dagger - a^\dagger a = 1 \Longrightarrow aa^\dagger = 1 + a^\dagger a$

$$C = \left(H^Y\right)^2 = V_0^2 \cdot 2eB \begin{pmatrix} a^\dagger a & 0 \\ 0 & aa^\dagger \end{pmatrix} = V_0^2 \cdot 2eB \begin{pmatrix} a^\dagger a & 0 \\ 0 & a^\dagger a + 1 \end{pmatrix}$$

Let us define: $N = a^{\dagger}a$ so that $N \mid n >= n \mid n > \Longrightarrow$
$C = V_0^2 \cdot 2eB \begin{pmatrix} n & 0 \\ 0 & n+1 \end{pmatrix}$
$u_n = \begin{pmatrix} n \\ 0 \end{pmatrix}$, $u_{n+1} = \begin{pmatrix} 0 \\ n+1 \end{pmatrix}$, $\lambda_1 = n$, $\lambda_2 = n+1$
$E_n^2 = V_0^2 \cdot 2eBn$, $\lambda_n = n = 0, 1, 2....$ and E_n is of degeneracy 2 for every n but $n = 0$

(4) $H^Y = V_0 \cdot \sqrt{2eB} \begin{pmatrix} 0 & a \\ a^{\dagger} & 0 \end{pmatrix}$, $C = \left(H^Y\right)^2 = V_0^2 \cdot 2eB \begin{pmatrix} a^{\dagger}a & 0 \\ 0 & a^{\dagger}a+1 \end{pmatrix}$ so $[H, C] = 0$
so C and H share the same eigen states;
$u_1 = \begin{pmatrix} \mid n > \\ 0 \end{pmatrix}$, $u_2 = \begin{pmatrix} 0 \\ \mid n+1 > \end{pmatrix}$
we can rearrange the order of the base elements so that
$C = \left(H^Y\right)^2 = V_0^2 \cdot 2eB \begin{pmatrix} n & 0 \\ 0 & n+1-1 \end{pmatrix} = V_0^2 \cdot 2eB \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}$
$u_1 = \begin{pmatrix} \mid n > \\ 0 \end{pmatrix}$, $u_2 = \begin{pmatrix} 0 \\ \mid n > \end{pmatrix}$
$C = \left(H^Y\right)^2 \Longrightarrow H^Y = \sqrt{C} = v_0\sqrt{2eB} \begin{pmatrix} 0 & \sqrt{n} \\ \sqrt{n} & 0 \end{pmatrix}$
we can rearrange the order of the base elements so that
$\sqrt{C} = H^{Y,n} = v_0\sqrt{2eB} \begin{pmatrix} 0 & \sqrt{n} \\ \sqrt{n} & 0 \end{pmatrix} \Longrightarrow$

(5) $H^{Y,n} = V_0 \cdot \sqrt{2eBn} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $E_{Y,n} = \pm V_0 \cdot \sqrt{2eBn}$

(6) In the first form of the Y blocks, the eigen vectors of the matrix:
$H = v_0\sqrt{2eB} \begin{pmatrix} 0 & a^{\dagger} \\ a & 0 \end{pmatrix}$ is $u_n^+ = \begin{pmatrix} \mid n > \\ \mid n+1 > \end{pmatrix}$, $u_n^- = \begin{pmatrix} \mid n > \\ - \mid n+1 > \end{pmatrix}$
Using Q and P we see that the given Hemiltonian is analogous to one of a harmonic oscilator, therefore we can rewrite the eigenvectors in terms of the eigen functions of a harmonic oscilator;
$u_n^+ = \begin{pmatrix} \mid \varphi^n(y) > \\ \mid \varphi^{n-1}(y) > \end{pmatrix}$, $u_n^- = \begin{pmatrix} \mid \varphi^n(y) > \\ - \mid \varphi^{n-1}(y) > \end{pmatrix}$
Though since the oscilator is shifted we need to add a phase factor of the form e^{-ieBYx} so the eigen states of the given Hemiltonian will be:
$\overline{u_n^+} = \begin{pmatrix} \mid \varphi^n(y) > \\ \mid \varphi^{n-1}(y) > \end{pmatrix} e^{-ieBYx}$, $\overline{u_n^-} = \begin{pmatrix} \mid \varphi^n(y) > \\ - \mid \varphi^{n-1}(y) > \end{pmatrix} e^{-ieBYx}$
* if we concider the eigen functions to be normlized, another normlizing factor should be added