Ex4853: Dynamics Of Two Coupled Spins

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The problem:

Two coupled spins are described by the Hamiltonian, $\mathcal{H} = \mu \mathbf{S}^A \cdot \mathbf{S}^B + hS_z^A$, when $\mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}$. The first spin was prepared in Z polarization. We will focus on the movemnt of the second spin. The spin state is described by a polarization vector $\mathbf{M} = (\langle\sigma_x^B\rangle, \langle\sigma_y^B\rangle, \langle\sigma_z^B\rangle)$.

(1) Write the Hamiltonian of the system in the standard basis.

The second spin is prepared in X polarization.

(2) What will be his frequency in the limit $h \to \infty$?
(3) What will be his frequency in the limit $h \to 0$?

Now the second spin is prepared in $-Z$ polarization (spin down).

(4) What will be his frequency ? $\mu,\ h$ are given.
(5) Write an expression for the polarization vector $\mathbf{M}(t)$.
(6) Write a condition so that for all times $\mathbf{M}(t) \neq 0$.

In section 5 it is possible to use known expressions for spin precession (Rabi formula).

The solution:

(1) Our problem is 4 dimensional, with the standard basis $|m^A\rangle \otimes |m^B\rangle$.

$$|\uparrow\uparrow\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} ;\ |\uparrow\downarrow\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} ;\ |\downarrow\uparrow\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} ;\ |\downarrow\downarrow\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

To find the Hamiltonian in the standard basis it's usefull to rewrite it as:

$$\mathcal{H} = \frac{\mu}{2}((\mathbf{S}^A + \mathbf{S}^B)^2 - (\mathbf{S}^B)^2 - (\mathbf{S}^A)^2) + hS_z^A$$

which is more easy to handl, since for $(\mathbf{S}^A + \mathbf{S}^B)^2 = \mathbf{J}^2$ we know the matrix (lecture notes 36.1), and $\left(\mathbf{S}^B\right)^2$, $\left(\mathbf{S}^A\right)^2$, are still diagonal in this basis. in this basis the Hamiltonian is:

$$\mathcal{H} = \frac{\mu}{4} \cdot \begin{pmatrix}1 & & & \\ & -1 & 2 & \\ & 2 & -1 & \\ & & & 1\end{pmatrix} + \frac{h}{2} \cdot \begin{pmatrix}1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1\end{pmatrix}$$

(2) Because the spin of A is up, when $h \to \infty$, it will not change during the system evolution. That means that all the evolution is occuring in the 2×2 subspace of $|\uparrow, m\rangle$. If we will set the energy zero to $\frac{h}{2}$ the effective Hamiltonian will be:

$$\mathcal{H} = \frac{\mu}{4} \cdot \begin{pmatrix}1 & \\ & -1\end{pmatrix}$$

which can be written as $\mathcal{H} = \frac{\mu}{2} \cdot S_z^B$. Any two levels system Hamiltonian can be put in the form $\mathcal{H} = \mathbf{\Omega} \cdot \mathbf{S}$, so by comparsion to the latter expression we find that:

$$\mathbf{\Omega} = \left(0, 0, \frac{\mu}{2}\right) ; \; |\mathbf{\Omega}| = \frac{\mu}{2}$$

(3) By observing the Hamiltonian matrix form we find that only two states are coupled $|\uparrow, \downarrow\rangle$, $|\downarrow, \uparrow\rangle$, we can write the effective Hamiltonian for this subspace as:

$$\mathcal{H} = -\mathbb{I} + \mu \cdot S_x$$

notice that here S_x is a 2×2 matrix defined on our subspace. The system frequency will be: $|\mathbf{\Omega}| = \mu$

(4) From the Hamiltonian it is clear that the initial state of the system $|\uparrow\downarrow\rangle$ is still coupled only to $|\downarrow\uparrow\rangle$, so considering the subspace of these two states the Hamiltonian is now (after redefining the energy floor):

$$\mathcal{H} = \mu S_x + h S_z$$

The frequency of the system is $\mathbf{\Omega} = (\mu, 0, h) ; \; |\mathbf{\Omega}| = \sqrt{\mu^2 + h^2}$.

(5) We are dealing with a two level system so the time evolution can be written with the help of Rabi formula for the survival probabilty $P(t) = 1 - \sin^2(\theta)\sin^2(\frac{\Omega t}{2})$, when $\theta = \arctan(\frac{\mu}{h})$, and Ω is the system frequncy that was found in the previuos section:

$$|\psi_0\rangle = |\uparrow\downarrow\rangle \implies |\psi_t\rangle = P(t)|\uparrow\downarrow\rangle + (1 - P(t))|\downarrow\uparrow\rangle$$

Now we can calculate the polarization vector in the usual way. we can simplify the algebra by observing that by definition $\boldsymbol{\sigma}^B$ is acting only on $\left|m^B\right\rangle$:

$$\sigma_x^B|\psi_t\rangle = P(t)|\uparrow\uparrow\rangle + (1 - P(t))|\downarrow\downarrow\rangle$$

so obviously:

$$\langle\psi_t|\sigma_x^B|\psi_t\rangle = 0$$

with similar calculation one find's that $\langle\psi_t|\sigma_y^B|\psi_t\rangle = 0$. for σ_z^B:

$$\sigma_z^B|\psi_t\rangle = -P(t)|\uparrow\downarrow\rangle + (1 - P(t))|\downarrow\uparrow\rangle$$

so:

$$\langle\psi_t|\sigma_z^B|\psi_t\rangle = (1 - P(t))^2 - P(t)^2 = 1 - 2P(t)$$

so the polarization vector is:

$$\mathbf{M}(t) = (0, 0, 1 - 2P(t))$$

(6) In order of the spin not to get to 0 we demand that the polariztion vector wil not get null which means:

$$1 - 2P(t) < 0 \implies P(t) > \frac{1}{2}$$

this is true because $M_z(0) = -1$ and it changes continuosly. this condition implies that:

$$1 - \sin^2(\theta)\sin^2(\frac{\Omega t}{2}) > \frac{1}{2}$$

and because $\sin^2(\frac{\Omega t}{2}) \in \{0, 1\}$ this means that:

$$|\sin(\theta)| < \frac{1}{\sqrt{2}}; |\theta| < \frac{\pi}{4}$$

inserting the expression for θ the condition is:

$$|\arctan(\frac{\mu}{h})| < \frac{\pi}{4}$$

wihch means:

$$\left|\frac{\mu}{h}\right| < 1 \implies |\mu| < |h|$$