E485: Entangled states of two spin-1/2 particles

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The problem:

For a system of 2 particles with spin 1/2 , the spin states of the system are presented in the standard basis: $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$.
The system was prepared in the state: $|\psi\rangle \propto |\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$.

Notice that the system was prepared in a state which is symmetric under permutation, so both particles have the same polarization state.

(1) Find the representation of the S_x operator, which describes the spin of one of the particles, in matrix form in the standard basis described above.
(2) Find the polarization vector, $\overrightarrow{M} = (\langle S_x\rangle, \langle S_y\rangle, \langle S_z\rangle)$, for each one of the particles.
(3) Write the polarization direction $\hat{n}$ and the relative polarization size $p = \frac{|M|}{M_{max}}$ for each one of the particles.
(4) What is the state vector in the *up/down* basis, which describes a single spin with full polarization in $\hat{n}$ direction.
(5) What will be the system state if the 2 particles are prepared at a state of full polarization in $\hat{n}$ direction (in the standard basis described above).

The solution:

If not sited otherwise, all the solution are presented in the standard basis:

$$|\uparrow\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |\uparrow\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, |\downarrow\uparrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, |\downarrow\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and the system state is $|\psi\rangle = |S_a S_b\rangle$ when S_i is the spin of the i-th particle.
In addition, talking into account that the state given in this question is symmetric under permutations, when needed, without loss of generality we will answer for particle 'a' and the answer for particle 'b' will be the same.

(1) Expanding the S_x operator to our standard basis, to find the x component of the spin of particle 'a':

$$S_x^a = S_x \otimes I_{2x2} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

To find the x component of the spin of particle 'b':

$$S_x^b = I_{2x2} \otimes S_x = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(2) To find $\overrightarrow{M} = (\langle S_x^a \rangle, \langle S_y^a \rangle, \langle S_z^a \rangle)$, the polarization vector for particle 'a' we need to find first the representation of the three polarization operators for 'a':

$$S_x^a = S_x \otimes I_{2x2} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad S_y^a = S_y \otimes I_{2x2} = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$S_z^a = S_z \otimes I_{2x2} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Now for the state given, after normalizing, $|\psi\rangle = \frac{1}{\sqrt{3}}(|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix}^T$, we will calculate the polarization:

$$\langle S_x^a \rangle = \langle \psi | S_x^a | \psi \rangle = \frac{1}{6}\begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{3}$$

$$\langle S_y^a \rangle = \langle \psi | S_y^a | \psi \rangle = \frac{1}{6}\begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$\langle S_z^a \rangle = \langle \psi | S_z^a | \psi \rangle = \frac{1}{6}\begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{6}$$

giving $\overrightarrow{M} = \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{6} \end{pmatrix}$ for particle 'a', and we will get the same result for particle 'b'.

(3) Given the polarization vector $\overrightarrow{M} = \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{6} \end{pmatrix}$ found before, the polarization direction is $\hat{M} = \frac{\overrightarrow{M}}{M} = \frac{\begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{6} \end{pmatrix}}{\frac{\sqrt{5}}{6}} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 0 & 1 \end{pmatrix}$ calculating angles give $sin(\theta) = \frac{2}{\sqrt{5}}$, $cos(\theta) = \frac{1}{\sqrt{5}}$, $\phi = 0$.

Preparing the system with maximum polarization, for example in the state $|\uparrow\uparrow\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^T$, will give $\overrightarrow{M} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \end{pmatrix}$ so $M_{max} = \frac{1}{2}$. Hence $p = \frac{|M|}{M_{max}} = \frac{\sqrt{5}}{3}$.

We notice that we got a polarization vector smaller in size than 1 for each particle. This is due to the nature of the state given in this question which is an entangled state, in which each spin in separate in not fully polarized.

(4) In this section we are working in the normal *up/down* basis of one spin $\frac{1}{2}$.

In order to get a single spin fully polarized in $\hat{n}$ direction, we will take a spin pointing up $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and rotate it to the desired direction. Remembering that $\phi = 0$, we will rotate the spin-up by θ

around y-axis:

$$|\xi\rangle = R^y_{(\theta)}\,|\uparrow\rangle = \begin{pmatrix} cos(\frac{\theta}{2}) & -sin(\frac{\theta}{2}) \\ sin(\frac{\theta}{2}) & cos(\frac{\theta}{2}) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} cos(\frac{\theta}{2}) \\ sin(\frac{\theta}{2}) \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1+cos(\theta)}{2}} \\ \sqrt{\frac{1-cos(\theta)}{2}} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1+\frac{1}{\sqrt{5}}}{2}} \\ \sqrt{\frac{1-\frac{1}{\sqrt{5}}}{2}} \end{pmatrix}$$

(5) We prepare each one of the particles as described above to get them fully polarized in the desired direction, hence $|\xi_a\rangle = |\xi_b\rangle = |\xi\rangle$. Now to get the system state of the 2 spins together in our standard basis:

$$\left|\tilde{\psi}\right\rangle = |\xi_a\rangle \otimes |\xi_b\rangle = \begin{pmatrix} \sqrt{\frac{1+\frac{1}{\sqrt{5}}}{2}} \\ \sqrt{\frac{1-\frac{1}{\sqrt{5}}}{2}} \end{pmatrix} \otimes \begin{pmatrix} \sqrt{\frac{1+\frac{1}{\sqrt{5}}}{2}} \\ \sqrt{\frac{1-\frac{1}{\sqrt{5}}}{2}} \end{pmatrix} = \begin{pmatrix} \frac{1+\frac{1}{\sqrt{5}}}{2} \\ \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ \frac{1-\frac{1}{\sqrt{5}}}{2} \end{pmatrix}$$

The result of preparing each spin separately pointing in $\hat{n}$ and combining them - $\left|\tilde{\psi}\right\rangle$, is different from $|\psi\rangle$ which describes the two spins system when the polarization of each spin was measured also in $\hat{n}$. Measuring the polarization of one of the spins out of $\left|\tilde{\psi}\right\rangle$will give us $\frac{1}{2}$, unlike in $|\psi\rangle$ where each spin's polarization is less than $\frac{1}{2}$. This is due to the nature of the entangled state $|\psi\rangle$.