E485: Interwoven states of 2 spins

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The problem:

There are 2 particles of spin $\frac{1}{2}$. The spin states of the system are being describe in the standart base:

$|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$

The system has being prepered in the state: $|\psi\rangle \propto |\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$
Notice: The system has being prepered in a symertic state compere to the pormutation, so the tw particles has the same polarization state.

(1) Find the expration of the operator S_x, which describe the spin of one of the particles as a matric in the standart base, which has defined abouve.

(2) Find the polarization vector $\vec{M} = (\langle S_x\rangle, \langle S_y\rangle, \langle S_z\rangle)$ which describe the polarization state of each particle.

(3) Write the polarization direction $\vec{n}$ and polarization rate $p = \frac{|\vec{M}|}{M_{max}}$ of the state of one particle.

(4) What is the state vector in the base: *up/down* , which describes single spin with full polarization in $\vec{n}$ direction.

(5) What will be the system state if the 2 particles are being prepered at a state of full polarization, in $\vec{n}$ direction.

The solution:

(1) The standard base is: $|\uparrow\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |\uparrow\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, |\downarrow\uparrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, |\downarrow\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$

The system is defined by: $\Psi = |S_a, S_b\rangle$.

In order to write the operator S_x, which describe the spin of one of the particles, we need to write it, in the standart base witen abouve.

For the first particls: $S_x^1 = S_x \otimes \hat{I} = S_x = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix};$

And, for the second particle: $S_x^2 = \hat{I} \otimes S_x = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$

(2) We are being asked to find $\vec{M} = (\langle S_x\rangle, \langle S_y\rangle, \langle S_z\rangle)$ for each particle.

For each one of the particles $j = 1, 2$ we define:

$\langle S_i^j\rangle = \langle\Psi|\hat{S_i^j}|\Psi\rangle$, where $i = x, y, z$ and $j = 1, 2$ (the particle index)

For the **First** particle ($j = 1$): let's find the operators $\hat{S_i^1}$:

When we define:

$$S_x^1 = \frac{1}{2}\sigma_x \otimes \hat{I}\ , \quad S_y^1 = \frac{1}{2}\sigma_y \otimes \hat{I}\ , \quad S_z^1 = \frac{1}{2}\sigma_z \otimes \hat{I}$$

we obtain the matrices S_x^1, S_y^1, S_z^1 in the standart basis,

$$S_x^1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad S_y^1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -\imath & 0 \\ 0 & 0 & 0 & -\imath \\ \imath & 0 & 0 & 0 \\ 0 & \imath & 0 & 0 \end{pmatrix}, \quad S_z^1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Our wave function is given: $\Psi = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

so:

$$\langle S_x^1\rangle = \frac{1}{6}\begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{3}$$

$$\langle S_y^1\rangle = \frac{1}{6}\begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -\imath & 0 \\ 0 & 0 & 0 & -\imath \\ \imath & 0 & 0 & 0 \\ 0 & \imath & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$\langle S_z^1\rangle = \frac{1}{6}\begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{6}$$

$$\vec{M}^1 = \frac{1}{6}(2, 0, 1)$$

For the **Second** particle ($j = 2$): let's find the operators $\hat{S_i^2}$:

When we define:

$$S_x^2 = \hat{I} \otimes \frac{1}{2}\sigma_x \ , \quad S_y^2 = \hat{I} \otimes \frac{1}{2}\sigma_y \ , \quad S_z^2 = \hat{I} \otimes \frac{1}{2}\sigma_z$$

we obtain the matrices S_x^2, S_y^2, S_z^2 in the standart basis,

$$S_x^2 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad S_y^2 = \frac{1}{2}\begin{pmatrix} 0 & -\imath & 0 & 0 \\ \imath & 0 & 0 & 0 \\ 0 & 0 & 0 & -\imath \\ 0 & 0 & \imath & 0 \end{pmatrix}, \quad S_z^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

so:

$$\langle S_x^2 \rangle = \frac{1}{6}\begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{3}$$

$$\langle S_y^2 \rangle = \frac{1}{6}\begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\imath & 0 & 0 \\ \imath & 0 & 0 & 0 \\ 0 & 0 & 0 & -\imath \\ 0 & 0 & \imath & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$\langle S_z^2 \rangle = \frac{1}{6}\begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}$$

$$\vec{M}^2 = \frac{1}{6}(2, 0, 3)$$

(3) Now we need to find the polarization direction $\vec{n}$ and polarization rate $p = \frac{|\vec{M}|}{M_{max}}$.

If a spin is completely polorized so, it point to a spesifice direction. in case of spine half the maximum Amplitude of this vector is 1/2 , as a result of stern-gerlach experience $\Rightarrow M_{max} = 1/2$, (for spin one we get: $M_{max} = 1$)

So, for the **First** spin we get:

$$|\vec{M}^1| = \frac{\sqrt{5}}{6}, \quad \Rightarrow \quad \vec{n^1} = \frac{\vec{M}}{|\vec{M}|} = \frac{1}{\sqrt{5}}(2, 0, 1); \quad p^1 = \frac{|\vec{M}|}{M_{max}} = \frac{\sqrt{5}}{3}$$

And, for the **Second** spin we get:

$$|\vec{M}^2| = \frac{\sqrt{13}}{6}, \quad \Rightarrow \quad \vec{n^2} = \frac{\vec{M}}{|\vec{M}|} = \frac{1}{\sqrt{13}}(2, 0, 3); \quad p^2 = \frac{|\vec{M}|}{M_{max}} = \frac{\sqrt{13}}{3}$$

(4) What is the state vector in the base: *up/down*?
The base *up/down* is:

$$|\uparrow\rangle \mapsto \begin{pmatrix}1\\0\end{pmatrix} \quad |\downarrow\rangle \mapsto \begin{pmatrix}0\\1\end{pmatrix}$$

we will take spin *up* which is $\begin{pmatrix}1\\0\end{pmatrix}$ and spin it around y ases in θ:

$$\begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix}$$

For the **First** case:

$$\hat{n}_1 = \frac{1}{\sqrt{5}}(2,0,1) \quad \text{Therefore:}$$

$$\cos\theta = \frac{1}{\sqrt{5}}, \quad \sin\theta = \frac{2}{\sqrt{5}} \Rightarrow$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}} = \pm\sqrt{\frac{1+\frac{1}{\sqrt{5}}}{2}}, \quad \sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} = \pm\sqrt{\frac{1-\frac{1}{\sqrt{5}}}{2}}$$

$$\text{while} \quad \theta_1 = \arctan(2)$$

Hence:

$$\hat{\psi}_1 = \begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix} = \begin{pmatrix}\sqrt{\frac{1+\frac{1}{\sqrt{5}}}{2}}\\ \sqrt{\frac{1-\frac{1}{\sqrt{5}}}{2}}\end{pmatrix}$$

For the **Second** case:

$$\hat{n}_2 = \frac{1}{\sqrt{13}}(2,0,3) \quad \text{Therefore:}$$

$$\cos\theta = \frac{3}{\sqrt{13}}, \quad \text{and} \quad \sin\theta = \frac{2}{\sqrt{13}} \Rightarrow$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}} = \pm\sqrt{\frac{1+\frac{3}{\sqrt{13}}}{2}} \quad \sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} = \pm\sqrt{\frac{1-\frac{3}{\sqrt{13}}}{2}}$$

$$\text{while} \quad \theta_2 = \arctan(\frac{2}{3})$$

Hence:

$$\hat{\psi}_2 = \begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix} = \begin{pmatrix}\sqrt{\frac{1+\frac{3}{\sqrt{13}}}{2}}\\ \sqrt{\frac{1-\frac{3}{\sqrt{13}}}{2}}\end{pmatrix}$$

(5) What will be the system state if the 2 particles are being prepered at: (a) $\hat{n_1}$, (b) $\hat{n_2}$? Here there are 2 particles. The wave function will be tensor product between them:

$$|\Psi\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} \otimes \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \cos^2\frac{\theta}{2} \\ \frac{1}{2}\sin\theta \\ \frac{1}{2}\sin\theta \\ \sin^2\frac{\theta}{2} \end{pmatrix}$$

So, For **(a)** we will get: $|\Psi_a\rangle = \begin{pmatrix} \cos^2\frac{\theta}{2} \\ \frac{1}{2}\sin\theta \\ \frac{1}{2}\sin\theta \\ \sin^2\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(1+\frac{1}{\sqrt{5}}\right) \\ \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ \frac{1}{2}\left(1-\frac{1}{\sqrt{5}}\right) \end{pmatrix}$

And, For **(b)** we will get: $|\Psi_b\rangle = \begin{pmatrix} \frac{1}{2}\left(1+\frac{3}{\sqrt{13}}\right) \\ \frac{1}{\sqrt{13}} \\ \frac{1}{\sqrt{13}} \\ \frac{1}{2}\left(1-\frac{3}{\sqrt{13}}\right) \end{pmatrix}$

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Remarks:


All the results are the same for both particles.
The results for the "first" particle are correct
and the results for the "second" particle are wrong.
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