E484: Singlet and Triplet States

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The problem:

The system of two spins is found to be in the singlet or one of the triplet states. What is a quantum state of measured spin (polarization vector) in every considered state of the system?

The solution:

Let us choose basis of considered states in following way

$$|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$$

Then one can define the states in this basis as follows,

$$\text{Singlet} \Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \qquad \text{Triplet} \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Using the Pauli matrices,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we obtain the matrices S_{1x}, S_{1y}, S_{1z} in the $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$ basis,

$$S_{1x} = \frac{\hbar}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad S_{1y} = \frac{\hbar}{2}\begin{pmatrix} 0 & 0 & -\imath & 0 \\ 0 & 0 & 0 & -\imath \\ \imath & 0 & 0 & 0 \\ 0 & \imath & 0 & 0 \end{pmatrix}, \quad S_{1z} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

When we define:

$$S_{1x} = \frac{\hbar}{2}\sigma_x \otimes \hat{I}\ , \quad S_{1y} = \frac{\hbar}{2}\sigma_y \otimes \hat{I}\ , \quad S_{1z} = \frac{\hbar}{2}\sigma_z \otimes \hat{I}$$

Since the polarization vector of the measured spin is

$$\vec{M} = (\langle S_{1x}\rangle, \langle S_{1y}\rangle, \langle S_{1z}\rangle)$$

we can calculate mean values of the matrices S_{1x}, S_{1y}, S_{1z} in every considered state of the system.

For singlet state $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$:

$$\langle S_{1x}\rangle = \frac{\hbar}{4}\begin{pmatrix} 0 & 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = 0$$

$$\langle S_{1y}\rangle = \frac{\hbar}{4}\begin{pmatrix} 0 & 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -\imath & 0 \\ 0 & 0 & 0 & -\imath \\ \imath & 0 & 0 & 0 \\ 0 & \imath & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = 0$$

$$\langle S_{1z}\rangle = \frac{\hbar}{4}\begin{pmatrix} 0 & 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = 0$$

$$\vec{M} = (0, 0, 0)$$

For triplet state $|\uparrow\uparrow\rangle$:

$$\langle S_{1x}\rangle = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$\langle S_{1y}\rangle = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -\imath & 0 \\ 0 & 0 & 0 & -\imath \\ \imath & 0 & 0 & 0 \\ 0 & \imath & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$\langle S_{1z}\rangle = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{\hbar}{2}$$

$$\vec{M} = (0, 0, \frac{\hbar}{2})$$

For triplet state $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$:

$$\langle S_{1x}\rangle = \frac{\hbar}{4}\begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$\langle S_{1y}\rangle = \frac{\hbar}{4}\begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -\imath & 0 \\ 0 & 0 & 0 & -\imath \\ \imath & 0 & 0 & 0 \\ 0 & \imath & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$\langle S_{1z}\rangle = \frac{\hbar}{4}(\ 0 \quad 1 \quad 1 \quad 0\ ) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$\vec{M} = (0, 0, 0)$$

For triplet state $|\downarrow\downarrow\rangle$:

$$\langle S_{1x}\rangle = \frac{\hbar}{2}(\ 0 \quad 0 \quad 0 \quad 1\ ) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$\langle S_{1y}\rangle = \frac{\hbar}{2}(\ 0 \quad 0 \quad 0 \quad 1\ ) \begin{pmatrix} 0 & 0 & -\imath & 0 \\ 0 & 0 & 0 & -\imath \\ \imath & 0 & 0 & 0 \\ 0 & \imath & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$\langle S_{1z}\rangle = \frac{\hbar}{2}(\ 0 \quad 0 \quad 0 \quad 1\ ) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = -\frac{\hbar}{2}$$

$$\vec{M} = (0, 0, -\frac{\hbar}{2})$$

We could also look on the second particle. Using the same Pauli matrices we obtain:

$$S_{2x} = \frac{\hbar}{2}\hat{I} \otimes \sigma_x\ , \quad S_{2y} = \frac{\hbar}{2}\hat{I} \otimes \sigma_y\ , \quad S_{2z} = \frac{\hbar}{2}\hat{I} \otimes \sigma_z$$

$$S_{2x} = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad S_{2y} = \frac{\hbar}{2} \begin{pmatrix} 0 & -\imath & 0 & 0 \\ \imath & 0 & 0 & 0 \\ 0 & 0 & 0 & -\imath \\ 0 & 0 & \imath & 0 \end{pmatrix}, \quad S_{2z} = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

In fact this case simpler since matrix that we obtain is block diagonal (as we can see). So from technical point of view the calculation is simpler if done on the second particle.