E4730: Particle on a spherical shell, dynamics

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The problem:

A particle with mass M and charge e is placed on a spherical shell with radius R. Note that the standard observation variables that describe the state of the particle are $(\theta, \varphi, L_x, L_y, L_z)$. In this question we assume that the particle can be excited from the ground state to the first energy level, but not beyond that, so the states space is four dimensional. Mark the states of the base with $|0\rangle, |\uparrow\rangle, |\updownarrow\rangle$ and $|\downarrow\rangle$.

(1) What is the energy of each of the base states. Define the ground state as zero-energy.
The particle is prepared as "concentrated" as possible at the north pole of the shell.
(2) Write the wave function $\psi(\theta, \varphi)$ of the particle.
The state you wrote is not stationary.
The particle oscillates between the "north pole" and the "south pole".
(3) What is the time period, T_0, of the oscillations?
First case: We add a constant electric field, ϵ, in the z direction.
(4) Write Formally the perturbation element using the observation variables.
(5) What is the time period, T, of the oscillations?
Second case: We add a constant magnetic field, B, in the x direction.
(6) Write Formally the perturbation element using the observation variables.
(7) Find the value of B so that in Time $T_0/2$ the particle will be in the initial state.

The solution:

(1) The Hamiltonian of a particle with mass M restricted to move on a spherical shell with radius R is given by:

$$\hat{H} = \frac{\hat{L}^2}{2MR^2}$$

The solutions for this Hamiltonian are the spherical harmonic functions. We know that the particle can be in the ground state or the first excited state, and that's why l=0,1. The eigenstates are:

$$|0\rangle = Y^{0,0}, |\updownarrow\rangle = Y^{1,0}, |\uparrow\rangle = Y^{1,1}, |\downarrow\rangle = Y^{1,-1}$$

and the energies are:

$$E_{l,m} = \frac{l(l+1)}{2MR^2}$$

In our case we get:

$$\hat{H}|0\rangle = 0 = E_{0,0}, \hat{H}|\updownarrow\rangle = \hat{H}|\uparrow\rangle = \hat{H}|\downarrow\rangle = \frac{1}{MR^2} = E_{1,0} = E_{1,1} = E_{1,-1}$$

(2) The particle is prepared as concentrated as possible at the "north pole" of the shell. That's why we should look for a function that is a superposition of the eigenfunctions and gets its maximum

absolute value at the "north pole". The only function that satisfies this condition is: $\psi = 1 + \cos\theta$. Choosing this function avoids the case that we would get if we choose $\psi = \cos\theta$, which would give us a maximum in both, the "north pole" and the "south pole". Let's express the wave function in terms of the eigenfunctions:

$$\psi = \sqrt{3}Y^{0,0} + Y^{1,0}$$

The normalized function is given by:

$$\psi = \frac{1}{2}\left(\sqrt{3}Y^{0,0} + Y^{1,0}\right)$$

(3) Let's express $\psi(t)$ using the evolution operator, taking in account that $E_{0,0} = 0, E_{1,0} = \frac{1}{MR^2}$:

$$\psi(t) = \frac{1}{2}\left(\sqrt{3}Y^{0,0} + Y^{1,0}e^{-i\frac{1}{MR^2}t}\right)$$

In order to find the period time of the oscillations, we will use the relation $\omega = E_{1,0} - E_{0,0} = \frac{1}{MR^2}$. Using the fact that $T = \frac{2\pi}{\omega}$, we get:

$$T_0 = 2\pi MR^2$$

(4) Adding an electric field in the z direction creates a potential $V = -e\epsilon R\cos\theta$. We want to find this perturbation matrix elements. Keeping in mind that $V = -e\epsilon R\sqrt{\frac{4\pi}{3}}Y^{1,0}$, we can realize that this perturbation couples only the eigenfunctions $Y^{0,0}, Y^{1,0}$, and all other couplings vanish. From calculation of this coupling we get:

$$\langle 0| V |\updownarrow\rangle = \langle\updownarrow| V |0\rangle = \int -Y^{0,0}e\epsilon R\sqrt{\frac{4\pi}{3}}Y^{1,0}Y^{1,0}d\Omega = -\frac{e\epsilon R}{\sqrt{3}}$$

(5) According to what we found, we can write the Hamiltonian matrix as follows:

$$\hat{H} = \begin{pmatrix} 0 & -\frac{e\epsilon R}{\sqrt{3}} & 0 & 0 \\ -\frac{e\epsilon R}{\sqrt{3}} & \frac{1}{MR^2} & 0 & 0 \\ 0 & 0 & \frac{1}{MR^2} & 0 \\ 0 & 0 & 0 & \frac{1}{MR^2} \end{pmatrix}$$

Examining only the subspace of the states $|0\rangle, |\updownarrow\rangle$we get:

$$\hat{H} = \begin{pmatrix} 0 & -\frac{e\epsilon R}{\sqrt{3}} \\ -\frac{e\epsilon R}{\sqrt{3}} & \frac{1}{MR^2} \end{pmatrix}$$

In order to express this Hamiltonian using Pauli matrices, we would like to gauge the Hamiltonian as follows:

$$\hat{H} = \begin{pmatrix} -\frac{1}{2MR^2} & -\frac{e\epsilon R}{\sqrt{3}} \\ -\frac{e\epsilon R}{\sqrt{3}} & \frac{1}{2MR^2} \end{pmatrix}$$

which yields:

$$\hat{H} = -\frac{e\epsilon R}{\sqrt{3}}\sigma_x - \frac{1}{2MR^2}\sigma_z$$

On the other hand, we know that $\hat{H} = \vec{\Omega} \cdot \vec{S}$ and $\vec{S} = \frac{1}{2}(\sigma_x, \sigma_y, \sigma_z)$, which yields:

$$\vec{\Omega} = \left(-\frac{2e\epsilon R}{\sqrt{3}}, 0, -\frac{1}{MR^2}\right)$$

In effect, the frequency of the oscillations is the norm of this vector:

$$\omega = \left|\vec{\Omega}\right| = \sqrt{\frac{4e^2\epsilon^2R^2}{3} + \frac{1}{M^2R^4}}$$

We recall that $T = \frac{2\pi}{\omega}$, and that's why

$$T = \frac{2\pi}{\sqrt{\frac{4e^2\epsilon^2R^2}{3} + \frac{1}{M^2R^4}}}$$

(6) Adding a magnetic field, while taking in account that this is a spinless particle, requires adding of the Zeeman term:

$$\hat{H}_{Zeeman} = -\frac{e}{2M}\vec{B} \cdot \vec{L}$$

Note that we neglect the diamagnetic term, because this is a second order perturbation.

Because the magnetic field is in the x direction, the perturbation term is:

$$\hat{V} = -\frac{e}{2M}B\hat{L}_x$$

(7) Now, the Hamiltonian is given by:

$$\hat{H} = \frac{\hat{L^2}}{2MR^2} - \frac{e}{2M}B\hat{L}_x$$

The term of the state $|0\rangle$ in the wave function is not affected by the operator $\hat{L}_x$, because it has a spherical symmetry. Therefore:

$$\psi(t) = e^{-i\frac{\hat{L^2}}{2MR^2}t}e^{i\frac{e}{2M}B\hat{L}_xt}\frac{1}{2}\left(\sqrt{3}\,|0\rangle + |\updownarrow\rangle\right) = \frac{1}{2}\left(\sqrt{3}\,|0\rangle + e^{-i\frac{1}{MR^2}t}e^{i\frac{e}{2M}B\hat{L}_xt}\,|\updownarrow\rangle\right)$$

We know that a rotation operator is given by $R = e^{-i\vec{L}\cdot\vec{\phi}}$and that $\phi = \Omega t$. That's why the term $e^{i\frac{e}{2M}B\hat{L}_x t}$ represents a rotation operator around the (-x) axis with $\Omega = \frac{eB}{2M}$.
It's easy to see that at $t = \frac{T_0}{2}$ the wave function is:

$$\psi\left(t = \frac{T_0}{2}\right) = \frac{1}{2}\left(\sqrt{3}\left|0\right\rangle - e^{i\frac{e}{2M}B\hat{L}_x t}\left|\updownarrow\right\rangle\right)$$

In order that at this time the wave function will return to it's initial state, we need that the rotation operator will rotate the state $\left|\updownarrow\right\rangle$ by 180 degrees around the (-x) axis.
Therefore:

$$\Omega\frac{T_0}{2} = \pi$$

After substituting we get:

$$\frac{eB}{2M}\frac{2\pi MR^2}{2} = \frac{\pi eR^2 B}{2} = \pi$$

Finally we get:

$$B = \frac{2}{eR^2}$$