E472: Motion on a spherical shell

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The problem:

Given a spherical shell with radius R ;and a particle with mass M and a charge e. In $t = -\infty$ the particle is in the ground state.

(1) Find the wave function: $\Psi(\theta, \varphi)$

Given an electric pulse:

$$\varepsilon(t) = \varepsilon_0 e^{-\frac{1}{2}(t/r)^2}$$

(2) Which energy levels can the particle pass to?
(3) What is the probability to find the particle not in the ground state?

The solution:

(1) The Hamiltonian of the particle with mass M restricted to move on a sphere with radius R (lets us assume for simplisity of the solution that R=1) is given by:

$$\hat{H}_0 = \frac{\hat{p}^2}{2M} = \frac{\hat{L}^2}{2MR^2} = \frac{\hat{L}^2}{2M}$$

the system is found at the ground state, therefore l=0 and the wave function of the system is:

$$\Psi(\mathrm{r}, \theta, \varphi) = F(R) Y_{00} \qquad E_{lm} = \frac{\hbar^2 l(l+1)}{2M}$$

(2) If the electric pulse is turn on the Hamiltonian of the system is parturbated by the $\hat{H}^{'}$:

$$\hat{H} = \hat{H}_0 + \hat{H}^{'}$$

$$\hat{H}^{'} = -q\varepsilon(t)\hat{z} = -q\varepsilon(t)\cos(\theta) = -q\varepsilon(t)\sqrt{\frac{4\pi}{3}} Y_{10}$$

the correction matrix is given by:

$$\left\langle 00 \middle| \hat{H}^{'} \middle| lm \right\rangle = -q\varepsilon_0 e^{-\frac{1}{2}(t/r)^2} \sqrt{\frac{4\pi}{3}} \int_0^{2\pi} d\varphi \int_0^{\pi} Y_{00} Y_{10} Y_{lm} \sin\theta d\theta$$

the integral vanishes for all $|lm\rangle$ except $|lm\rangle$=$|10\rangle$, that is to say that system can pass to the first excited level: (l=1, m=0)

$$\left\langle 00 \middle| \hat{H}^{'} \middle| 01 \right\rangle = \frac{-q\varepsilon(t)}{\sqrt{3}} \Rightarrow \hat{V} = \frac{-q\varepsilon_0 e^{-\frac{1}{2}(t/r)^2}}{\sqrt{3}} \delta_{l,1;m,0}$$

So, only $E_{00} \to E_{10}$, that is to say $0 \to \frac{\hbar^2}{M}$, transition is possible!

(3) The particle can be found only at two different states: $|00\rangle$ and $|10\rangle$, so the probability of the state $|10\rangle$ is:

$$P_{10} = \frac{1}{\hbar^2}\left|\int_{-\infty}^{\infty} V_{01} e^{i\Omega_{10}t} dt\right|^2 = \frac{1}{\hbar^2}\left|\int_{-\infty}^{\infty} \frac{-q\varepsilon_0 e^{-\frac{1}{2}(t/r)^2}}{\sqrt{3}} e^{i\Omega_{10}t} dt\right|^2 = \frac{(q\varepsilon_0)^2}{3\hbar^2}\left|\int_{-\infty}^{\infty} e^{-\frac{1}{2}(t/r)^2} e^{i\frac{\hbar}{M}t} dt\right|^2$$

in last transition we used:

$$\Omega_{10} = \frac{1}{\hbar}(E_1 - E_0) = \frac{\hbar}{M}$$

in order to solve the aforementioned integral one may use the solved integral (Schaum's handbook 15.75):

$$\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)} dx = \sqrt{\frac{\pi}{a}} e^{(b^2-4ac)/4a}$$

$$P_{10} = \frac{(q\varepsilon_0)^2}{3\hbar^2}(2\tau^2\pi)e^{-\frac{\hbar^2\tau^2}{M^2}}$$

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Do not assume that R=1.
The exercise shold be solved using known results from perturbation theory.
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