E4710: Motion on a spherical shell

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The problem:

Spherical shell with radius R and a particle with mass M and charge q. In this question you have to assume that the particle can be existed from the ground state ($l = 0$) to the first energy level ($l = 1$), but not beyond that, which means that the state space is four dimensional.
A particle is prepared as "concentrated" as possible in the "north pole" of the shell.

(1) Write the wave function $\Psi(\theta, \phi)$.
(2) Find the periodic time of the oscillations.
The polarization of the system is defined as $P(t) = \langle\Psi(t)|\hat{z}|\Psi(t)\rangle$
(3) Find the polarization as function of time $P(t)$.
Given: $Y_{00} = \frac{1}{\sqrt{4\pi}}$; $Y_{10}(\theta, \phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$; $Y_{1,\pm1} \propto \sin\theta$

The solution:

(1) We want that probability $|\Psi|^2$ to find the particle at north pole ($\theta = 0$) will be maximal, so we look for a solution of the following form $\Psi \sim 1 + \cos\theta$

$$\Psi = c(1 + \cos\theta) = c(\sqrt{3}Y_{00} + Y_{10})$$

$$|\Psi\rangle = c(\sqrt{3}|00\rangle + |10\rangle)$$

After Normalization: $\int |\langle\Psi|\Psi\rangle|^2 d\Omega = 1$ we find that $c = \frac{1}{2}$ and we get

$$\Psi = \frac{1}{2}(\sqrt{3}Y_{00} + Y_{10})$$

(2) The energy states of the particle of the shell $E_l = \frac{l(l+1)}{2MR^2}$. We are restricted to $l = 0, 1$ so the energies are $E_0 = 0$ and $E_1 = \frac{1}{MR^2}$. The time depended wave function is

$$\Psi(t) = \frac{\sqrt{3}}{2}\mathrm{e}^{-iE_0 t}Y_{00} + \frac{1}{2}\mathrm{e}^{-iE_1 t}Y_{10} = \frac{\sqrt{3}}{2}Y_{00} + \frac{1}{2}\mathrm{e}^{-i\frac{1}{MR^2}t}Y_{10}$$

we can see the function is periodic with frequency $\omega = E_1 = \frac{1}{MR^2}$, so the periodic time of oscillations is

$$T = 2\pi MR^2$$

(3) To find the polarization we use $z = Rcos\theta = \sqrt{\frac{3}{4\pi}}RY_{10}$

$$\langle\Psi|\cos\theta|\Psi\rangle = \langle\cos\theta\rangle = \left(\frac{\sqrt{3}}{2}\langle 00| + \frac{1}{2}\mathrm{e}^{i\omega t}\langle 10|\right)\cos\theta\left(\frac{\sqrt{3}}{2}|00\rangle + \frac{1}{2}\mathrm{e}^{-i\omega t}|10\rangle\right)$$

$$\cos\theta|00\rangle = \sqrt{\frac{4\pi}{3}}Y_{10}\frac{\sqrt{3}}{2}Y_{00} = \frac{1}{2}Y_{10} = |10\rangle$$

$$\langle\cos\theta\rangle = \alpha\langle 00|10\rangle + \frac{1}{4}\mathrm{e}^{i\omega t}\langle 10|10\rangle + \frac{1}{4}\mathrm{e}^{-i\omega t}\langle 10|10\rangle + \beta\langle 10|\cos\theta|10\rangle$$

the first and the last parts are zero and we get

$$\langle\cos\theta\rangle = \frac{1}{2}\cos(\omega t)$$

$$P(t) = \langle\Psi|\hat{z}|\Psi\rangle = \frac{1}{2}R\cos(\omega t)$$