Ex4640: Angular momentum of wave function

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The problem:

The wave function of a particle is $\psi(x,y,z) = f(r)z(z+1)$.

1. Write the wave function in $|r,\theta,\varphi\rangle$ representation.
2. Write the wave function in $|r,\ell,m\rangle$ representation.

   Given that:
   $Y^{00} = \sqrt{\frac{1}{4\pi}}$, $Y^{10} = \sqrt{\frac{3}{4\pi}}cos(\theta)$, $Y^{20} = \sqrt{\frac{5}{16\pi}}(3cos^2(\theta)-1)$,

   $M_n = \int_0^\infty r^n (f(r))^2 dr$

3. Write a condition on M_6 and M_4 so that the wave function will be normalized.

   Following it is given: $M_6 = \frac{5}{8\pi}$

4. What is a probability of the particle to be on spherical state ($\ell = 0$)?
5. What are the possible non-zero probability values of L_z?
6. What are the possible non-zero probability values of L_x?
7. Find the probability of the particle to be on polar state ($\ell = 1$) with $L_x = 1$.

The solution:

1. $\psi(r,\theta,\varphi) = f(r)r^2\cos^2\theta + f(r)r\cos\theta$
2. $\psi(r,\ell,m) = f(r)r^2\sqrt{\frac{16\pi}{45}}Y^{20} + f(r)r\sqrt{\frac{4\pi}{3}}Y^{10} + f(r)r^2\sqrt{\frac{4\pi}{9}}Y^{00}$
3. The normalization condition is

   $$\|\langle rm|rm\rangle\|^2 = (\frac{16\pi}{45}+\frac{4\pi}{9})\int_0^\infty r^6 f(r)dr + (\frac{4\pi}{3})\int_0^\infty r^4 f(r)dr = 1$$

   placing the definitions of M_n we get

   $$\frac{4\pi}{5}M_6 + \frac{4\pi}{3}M_4 = 1$$

4. $\|\langle r00|\psi\rangle\|^2 = \frac{4\pi}{9}M_6 = \frac{5}{18}$
5. $L_z = 0$ with probability 1.
6. I'll show that the possible values are $L_x = 0, \pm1, \pm2$:
   In order to find the possible values of L_x we'll go through the subspaces $\ell = 0,1,2$.

   - For $\ell = 0$ it is clear that we can get the value $L_x = 0$ from the possibility to get Y^{00}.
   - In the subspace $\ell = 1$ we know that

     $$Y^{10} = \frac{1}{\sqrt{2}}(Y_x^{11} - Y_x^{1-1})$$

     Hence we get that it is possible to get $L_x = \pm1$.

- The case of $\ell = 2$ is more complicated, therefore we sall prove by contridiction that it is possible to get $L_x = \pm 2$:
  Let us assume that the only possible values of L_x are $-1, 0, 1$. We know that $L^2 = L_x^2 + L_y^2 + L_z^2 = 2L_x^2$. Then we get $L^2 = 2$ is the maximum value. On the other hand we know that $\|\langle r20|\psi\rangle\|^2 > 0$ and in the $\ell = 2$ subspcae $L^2 = \ell(\ell+1) = 6$. Therefore we must conclude that the our first assumption is false and that the values $L_x = \pm 2$ are possible.

Q.E.D.

7. $\ell = 1$:

$$\|\langle L_x = 1|\psi\|^2 = \|\langle \vec{e_x}|\psi\|^2 = \frac{4\pi}{3} M_4 \|\langle \vec{e_x}| \Updownarrow \|^2 = \frac{1}{4}$$