Ex464: Angular momentum of wave function

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The problem:

The wave function of a particle is $\psi(x,y,z) = f(r)z(z+1)$.

(1) Write the wave function in $|r,\theta,\phi\rangle$ representation.
(2) Write the wave function in $|r,\ell,m\rangle$ representation.

Given that:
$Y^{00} = \sqrt{\frac{1}{4\pi}}$, $Y^{10} = \sqrt{\frac{3}{4\pi}}cos(\theta)$, $Y^{20} = \sqrt{\frac{5}{16\pi}}(3cos^2(\theta)-1)$,

$M_n = \int_0^\infty r^n (f(r))^2 dr$, $M_6 = \frac{5}{8\pi}$

(3) Write a condition on M_6 and M_4 so that the wave function will be normalized.
(4) What is a probability of the particle to be on spherical state ($\ell = 0$).
(5) What are the possible non-zero probability values of L_x?
(6) What are the possible non-zero probability values of L_z?
(7) Find the probability of the particle to be on polar state ($\ell = 1$) with $L_x = 1$.

The solution:

(1) Writing the wave function in spherical coordinates one gets:

$$\langle r,\theta,\phi|\Psi\rangle = f(r)r^2\cos^2(\theta) + f(r)r\cos(\theta) = \sqrt{\frac{16\pi}{45}}r^2 f(r)Y^{20}(\theta,\phi) - \sqrt{\frac{4\pi}{9}}r^2 f(r)Y^{00}(\theta,\phi) + \sqrt{\frac{4\pi}{3}}rf(r)Y^{10}(\theta,\phi)$$

(2)

$$\langle r,\ell,m|\Psi\rangle = \int \langle r,\ell,m|r',\theta,\phi\rangle\langle r',\theta,\phi|\Psi\rangle r'^2 dr' d\Omega = \sqrt{\frac{16\pi}{45}}r^3 f(r)\delta_{\ell 2}\delta_{m0} - \sqrt{\frac{4\pi}{9}}r^3 f(r)\delta_{\ell 0}\delta_{m0} + \sqrt{\frac{4\pi}{3}}r^2 f(r)\delta_{\ell 1}\delta_{m0}$$

Or in abstract form:

$$|\Psi\rangle = \sum_r (\psi_{20}(r)|r,2,0\rangle + \psi_{00}(r)|r,0,0\rangle + \psi_{10}(r)|r,1,0\rangle)$$

(3)

$$\langle\Psi|\Psi\rangle = \sum_{r,\ell,m} |\langle r,\ell,m|\Psi\rangle|^2 = \frac{16\pi}{45}M_6 + \frac{4\pi}{9}M_6 + \frac{4\pi}{3}M_4 = 1$$

By substituting the value of M_6 in the above expression one finds: $M_4 = \frac{3}{8\pi}$.

(4)

$$P = \sum_r |\langle r,0,0|\Psi\rangle|^2 = \frac{4\pi}{9}\int_0^\infty r^6 f^2(r)dr = \frac{4\pi}{9}M_6 = \frac{5}{18}$$

(5) This section will be solved by examining each sub-space ($\ell = 0, \ell = 1, \ell = 2$) separately.

In $\ell = 0$ sub-space:

It is trivial case when $L_x = 0$ is the only value.

In $\ell = 1$ sub-space:

Notations used:

Standart basis: $|m_x = -1\rangle, |m_x = 0\rangle, |m_x = 1\rangle$ **Linear basis:** $|m_x = 0\rangle, |m_y = 0\rangle, |m_z = 0\rangle$

$$|\Psi\rangle \propto |m_z = 0\rangle = \langle m_x = -1|m_z = 0\rangle|m_x = -1\rangle + \langle m_x = 1|m_z = 0\rangle|m_x = 1\rangle$$

If $\langle m_x = -1|m_z = 0\rangle = 0$ it would mean that $|m_z = 0\rangle \propto |m_x = 1\rangle$

If $\langle m_x = 1|m_z = 0\rangle = 0$ it would mean that $|m_z = 0\rangle \propto |m_x = -1\rangle$

If $\langle m_x = -1|m_z = 0\rangle = 0$ and $\langle m_x = 1|m_z = 0\rangle = 0$ it would mean that $|m_z = 0\rangle = 0$

All three cases above are impossible, hence $\langle m_x = -1|\Psi\rangle \neq 0$ and $\langle m_x = 1|\Psi\rangle \neq 0$, or in other words $L_x = \pm 1$ has non-zero probability.

In $\ell = 2$ sub-space:

$$\langle L^2\rangle = \langle L_x^2\rangle + \langle L_y^2\rangle + \langle L_z^2\rangle = \ell(\ell+1) = 6$$

Notice that $\langle L_z^2\rangle = 0$ because $\hat{L}_z|\Psi\rangle = 0|\Psi\rangle$ and $\langle L_x^2\rangle = \langle L_y^2\rangle$ because of symmetry considerations, therefore $\langle L^2\rangle = 2\langle L_x^2\rangle$, which implies that $\langle L_x^2\rangle = 3$.

Now suppose that getting $L_x = \pm 2$ has zero probability, then one would get:

$$\langle L_x^2\rangle = (-1)^2 P(L_x = -1) + 1^2 P(L_x = 1) < 3$$

Therefore values $L_x = \pm 2$ must have non-zero probabilities. Final conclusion implies that L_x can get values -2,-1,0,1,2.

(6)

$$\hat{L}_z|\Psi\rangle = \sum_r (\psi_{20}(r)\hat{L}_z|r,2,0\rangle + \psi_{00}(r)\hat{L}_z|r,0,0\rangle + \psi_{10}(r)\hat{L}_z|r,1,0\rangle) = 0|\Psi\rangle$$

(7)

$$\langle r,\ell=1,m_x=1|\Psi\rangle = \sum_{r',\ell',m_z} \langle r,\ell=1,m_x=1|r',\ell',m_z\rangle\langle r',\ell',m_z|\Psi\rangle = \psi_{10}(r)\langle m_x=1|m_z=0\rangle$$

One can express $\langle m_x = 1| = \frac{1}{2}(1,\sqrt{2},1)$ and $|m_z = 0\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}$, which implies $\langle m_x = 1|m_z = 0\rangle = \frac{1}{\sqrt{2}}$, also from section (2) it is clear that $\psi_{10}(r) = \sqrt{\frac{4\pi}{3}}r^2 f(r)$.

$$P = \sum_r^{\infty} |\langle r,\ell=1,m_x=1|\Psi\rangle|^2 = \int_0^{\infty} \frac{4\pi}{3} r^4 f^2(r)|\langle m_x=1|m_z=0\rangle|^2 dr = \frac{2\pi}{3}M_4 = \frac{1}{4}$$