E464: Angular momentum of wave function

Submitted by: Mor Yaakobi

The problem:

A particle is described by the wave function $\psi(x, y, z) = f(r)z(z + 1)$.

(1) Express the wave function in terms of $|r, \theta, \phi\rangle$.
(2) Express the wave function in terms of $|r, l, m\rangle$.
Given $Y_{00} = \sqrt{\frac{1}{4\pi}}$, $Y_{10} = \sqrt{\frac{3}{4\pi}} cos\theta$, $Y_{20} = \sqrt{\frac{5}{16\pi}}(3cos^2\theta - 1)$

Let us define $M_n = \int_0^\infty r^n(f(r))^2 dr$
(3) Write a condition for M_4 , M_6 so that the wave function will be normalized.
Given $M_6 = \frac{5}{8\pi}$
(4) What is the probability of finding the particle to be in a spherical state? $(l = 0)$
(5) What are the possible values for measuring L_x ?
(6) What are the possible values for measuring L_z ?
(7) What is the probability of finding the particle to be in a polar state $(l = 0)$, for $L_x = 1$?

The solution:

(1) Converting cartesian to spherical coordinates one gets

$$\psi(r, \theta, \phi) = f(r)r^2cos^2\theta + f(r)rcos\theta$$

(2) Using the orthonormal basis of the Hilbert space Y_{lm} one gets

$$\psi(r, l, m) = f(r)r^2(\sqrt{\frac{16\pi}{45}}Y^{20} + \sqrt{\frac{4\pi}{9}}Y^{00}) + f(r)r\sqrt{\frac{4\pi}{3}}Y^{10}$$

For simplicity, let us write the above expression schematically:

$$\psi(r, l, m) = f_0(r)Y^{00} + f_1(r)Y^{10} + f_2(r)Y^{20}$$

(3) Normalization requirement (using the schematic expression):

$$\langle\psi|\psi\rangle = \sum_{i=0}^{2} \int |f_i(r)|^2 r^2 dr = P_0 + P_1 + P_2 = \frac{4\pi}{9}M_6 + \frac{4\pi}{3}M_4 + \frac{16\pi}{45}M_6 = 1$$

(4) Let us sum over the probabilities of being in a spherical state $(l = 0)$

$$P = |\langle f_0(r)Y^{00}|\psi\rangle|^2 = P_0 = \frac{4\pi}{9}M_6 = \frac{5}{18}$$

(5) The possible values of L_x regarding the above wave function are $-2, -1, 0, 1, 2$. In order to find the values whose probability to be measured is bigger than 0, we will check if the aforementioned values are eigenvalues of L_x.

$$L_x = \frac{1}{2}(L_+ + L_-)$$

Let us examine each term of Y^{lm} separately:

Y^{00} is an eigenstate of L_x, with an eigenvalue of 0, thus is probable to be measured.
Apply L_x on the states $Y^{1,m}$, one gets the following eigenstates:

$$Y_x^{11} \mapsto \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix} \quad Y_x^{10} \mapsto \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ -1\end{pmatrix} \quad Y_x^{1-1} \mapsto \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -\sqrt{2}\\ 1\end{pmatrix}$$

with the eigenvalues of $1, 0, -1$ accordingly.

$$Y^{10} = \frac{1}{\sqrt{2}}(Y_x^{11} - Y_x^{1-1})$$

Thus $-1, 1$ are probable to be measured.
Remains find the representation of Y^{20} in the basis where L_x is diagonal. Using the same method, one gets

$$Y_x^{22} \mapsto \frac{1}{2}\begin{pmatrix}1\\ 2\\ \sqrt{6}\\ 2\\ 1\end{pmatrix} \quad Y_x^{20} \mapsto \frac{1}{2}\begin{pmatrix}1\\ 0\\ -\frac{2}{\sqrt{6}}\\ 0\\ 1\end{pmatrix} \quad Y_x^{2-2} \mapsto \frac{1}{2}\begin{pmatrix}1\\ -2\\ \sqrt{6}\\ -2\\ 1\end{pmatrix}$$

with the eigenvalues of $2, 0, -2$ accordingly.

$$Y^{20} = c_1 Y_x^{22} + c_2 Y_x^{22} + c_2 Y_x^{2-2}$$

Thus $-2, 2$ are also probable to be measured.

(6) Using r, l, m representation, L_z is determined to be diagonal. it follows that $L_z|m\rangle = |m\rangle$, and therefore the only possible value for measuring L_z is 0.

(7) The probability of measuring $L_x = 1$ when the particle is in a polar state ($l = 1$) is:

$$P = |\langle \Uparrow \mid \Updownarrow\rangle|^2 \times |\langle f_1(r) Y^{10}|\psi\rangle|^2 = |\langle \Uparrow \mid \Updownarrow\rangle|^2 \times P_1 = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$