Ex464: Angular momentum of a wave function

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The problem:

The wave function of a particle is $\psi(x, y, z) = f(r)z(z + 1)$.

(1) Write the wave function in the $|r\theta\phi\rangle$ basis.
(2) Write the wave function in the $|rlm\rangle$ basis.

given that:
$Y^{00} = \sqrt{\frac{1}{4\pi}}$, $Y^{10} = \sqrt{\frac{3}{4\pi}}cos(\theta)$, $Y^{20} = \sqrt{\frac{5}{16\pi}}(3cos^2(\theta) - 1)$

(3) write a condition on M_6 and M_4 so that the wave function will be normalized,Given $M_n = \int_0^\infty r^n(f(r))^2dr$ and $M_6 = \frac{5}{8\pi}$. find what is M_4.

(4) Find the probability of the particle to be in a spherical state (l=0).
(5) What are the possible values in the measurement of L_x?
(6) What are the possible values in the measurement of L_z?
(7) Find the probability of the particle to be in a polar state (l=1) with $L_x = 1$.

The solution:

(1) By changing variables from cartesian cordinates to spherical cordinates one gets:

$$\Psi(r, \theta, \phi) = f(r)r^2\cos^2(\theta) + f(r)r\cos(\theta)$$

(2) Using the Y^{lm} for the basis of Hilbert space one get:

$$\Psi(r, \theta, \phi) = f(r)r^2(\sqrt{\frac{16\pi}{45}}Y^{20} + \sqrt{\frac{4\pi}{9}}Y^{00}) + f(r)r\sqrt{\frac{4\pi}{3}}Y^{10}$$

Note by DC: Let us write it schemtically as

$$\Psi(r, \theta, \phi) = f_0(r)Y^{00} + f_1(r)Y^{10} + f_2(r)Y^{20}$$

(3) Normalizing the given wave function one get the wanted condition:

$$1 = \langle\Psi|\Psi\rangle = \int_{allspace}\Psi^*(r, \theta, \phi)\Psi(r, \theta, \phi)r^2drd\Omega = \frac{4\pi}{5}\int_0^\infty r^6(f(r))^2dr + \frac{4\pi}{3}\int_0^\infty r^4(f(r))^2dr = \frac{4\pi}{5}M_6 + \frac{4\pi}{3}M_4$$

If substitute M_6 in the above equation it is easy to find that: $M_4 = \frac{3}{8\pi}$.

Note by DC: Schematically the calculation here is

$$\text{Normalization} \;=\; \sum_{\ell=0}^{3}\int |f_\ell(r)|^2r^2dr \;=\; p_0 + p_1 + p_2 \;=\; \frac{4\pi}{9}M_6 + \frac{4\pi}{3}M_4 + \frac{16\pi}{45}M_6 \;=\; 1$$

(4) Finding the probability is as finding the projection of the given state on the wantet state to find, i.e in our case $\Phi = Y^{00}$. so we get:

$$P = |\langle \Phi | \Psi \rangle|^2 = \int_{allspace} |\Phi^*(\theta,\phi)\Psi(r,\theta,\phi)|^2 r^2 dr d\Omega = \frac{4\pi}{9} M_6 = \frac{5}{18}$$

Note by DC: The above way of writing is meaningless. what we have here is simply a sum over probabilities:

$$P = \sum_r |\langle r, \ell{=}0 | \Psi \rangle|^2 = p_0$$

(5) Using the same idea as in the last section we want to find the projection of our wave function on L_x. The given wave function consist three angular parts Y^{00},Y^{10} and Y^{20}. It is obvious that the Y^{00} can contribute only the value of $L_x = 0$ becouse l is equal to zero. Moving to the next term Y^{10}, we can express this term in the linear basis, so $Y^{10} = \frac{1}{\sqrt{2}}(-Y_x^{11} + Y_x^{1-1})$, from here we see that the possible values of L_x is -1 and 1. The last term is quite difficult to handle becouse it take a lot of algebra to express Y^{20} in another basis just as we did before,but we know that it cannot exceed the value of $l = 2$ and we allready got the values -1,0,1 so all we have to check is if it is possible to get also the values 2 and -2. We can proof that by negation (i.e "stira") assuming that the valuse 2 and -2 are not possible. We do know that on the one hand $\langle L^2 \rangle = 2(2+1) = 4$ while on the other hand $\langle L^2 \rangle = \langle L_x^2 \rangle + \langle L_y^2 \rangle + \langle L_z^2 \rangle$. Becouse of symmetry $\langle L_x^2 \rangle = \langle L_y^2 \rangle$ so we get $\langle L^2 \rangle = 2\langle L_x^2 \rangle$ which equel 4,in our problem $\langle L_z^2 \rangle = 0$, and from our assuming $\langle L_x^2 \rangle$ can get the values 0 or 1,therefore $\langle L^2 \rangle \leq 2$ in contra to $\langle L^2 \rangle = 4$ and we get a negation.

So the possible values in the system is $L_x = -2, -1, 0, 1, 2$.

Note by DC: The argumentation why for Y^{20} there is non-zero probability to measure $L_x = \pm 2$ is with typos. The correct phrasing is: For this state we have $\langle L^2 \rangle = (2+1)2 = 6$. On the other hand from XY symmetry consideration $\langle L^2 \rangle = \langle L_x^2 \rangle + \langle L_y^2 \rangle + \langle L_z^2 \rangle = 2\langle L_x^2 \rangle$. The latter implies $\langle L_x^2 \rangle = 3$, which cannot be true unless $L_x = \pm 2$ has non-zero probability.

(6) It is obviouse that the given wave function is an eigenvector of the operator L_z so we can see that the only Possible value that L_z can get is:

$$L_z = 0$$

(7) The probability of finding the particle in $l = 1$ and $L_x = 1$ is:

$$P(l=1, L_x=1) = |\langle l=1, L_x=1 | \Psi \rangle|^2 = |\langle l=1, L_x=1 | l=1, m=0 \rangle \langle l=1, m=0 | \Psi \rangle|^2 =$$

$$= |\langle \Uparrow | \Leftrightarrow \rangle|^2 \times \int |(Y^{10})^* \Psi|^2 d^3r = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Note by DC: The calculation here is

$$P = |\langle \Rightarrow | \Updownarrow \rangle|^2 \times p_1$$

and it is useful to observe that $|\langle \Rightarrow | \Updownarrow \rangle|^2 = |\langle \Uparrow | \Leftrightarrow \rangle|^2 = 1/2$, where the latter equality is implied by the standard representation of linear polarization along X.