# **Ex4622:** Precession caused by a series of pulses

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**The problem:**

A series of magnetic pulses are applied to a spin $\frac{1}{2}$ particle. The Hamiltonian is given by $\mathbf{h(t)}\cdot\mathbf{S}$. The duration of each pulse is $\tau$ and the amplitude is $h$. The field is in the $\hat{z}$ direction during the first pulse and in the $\hat{y}$ direction during the second pulse, and so on and so forth. As a result of these pulses we get a precession with frequency $\Omega$ about an axis $\mathbf{n}$.

(1) Write an expression for the precession frequency $\Omega$.
(2) Find the rotation axis $\mathbf{n}$ (unit vector).
(3) Find $\Omega \to \Omega_0$ in the limit where the driving force is a series of pulses with $\tau \to 0$ .
(4) Find $\mathbf{n} \to \mathbf{n_0}$ in the limit where the driving force is a series of pulses with $\tau \to 0$.
(5) Write an expression for $\tan\theta$ of the angle between $\mathbf{n}$ and $\mathbf{n_0}$.

Remarks:
The precession frequency is defined as $\Omega = \frac{\phi}{2\tau}$. Where $\phi$ is the equivalent rotation during two consecutive pulses. The answers to questions 3,4 are trivial and can be guessed. The expressions in question 2,5 after simplification include only a "tan" function.

**The solution:**

(1) The evolution operator from this Hamiltonian is a product of alternating rotations around the $\hat{y}$ and $\hat{z}$ axis. We can group each pair of consecutive rotations and examine their combined rotation.

$$\begin{aligned}
U(2\tau) = \mathrm{e}^{-ih\tau S_z}\mathrm{e}^{-ih\tau S_y} &= \left[\cos\left(\frac{h\tau}{2}\right)\cdot\hat{1} - i\sin\left(\frac{h\tau}{2}\right)\cdot\sigma_z\right]\cdot\left[\cos\left(\frac{h\tau}{2}\right)\cdot\hat{1} - i\sin\left(\frac{h\tau}{2}\right)\cdot\sigma_y\right] \\
&= \cos^2\left(\frac{h\tau}{2}\right)\cdot\hat{1} - i\left[\sin^2(\frac{h\tau}{2})\cdot\sigma_x + \cos(\frac{h\tau}{2})\sin(\frac{h\tau}{2})\cdot(\sigma_y+\sigma_z)\right] \\
&= \cos\left(\frac{\phi}{2}\right)\cdot\hat{1} - i\sin\left(\frac{\phi}{2}\right)\cdot\sigma_{\mathbf{n}}
\end{aligned}$$

We used the fact that any rotation can be expressed as a rotation with angle $\phi$ around some axis $\mathbf{n}$. Examining the coefficient for the unit matrix on both sides of the equation above yields:

$$\phi = 2\arccos\left(\cos^2\left(\frac{h\tau}{2}\right)\right) \to \Omega = \frac{\phi}{2\tau} = \frac{1}{\tau}\arccos\left(\cos^2\left(\frac{h\tau}{2}\right)\right)$$

(2) The rotation axis is similarly found from identifying $\hat{\mathbf{n}}$ in the left hand side of the equation above:

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{\tan^2(\frac{h\tau}{2})+2}}\left(\tan\left(\frac{h\tau}{2}\right),1,1\right)$$

(3) Small rotations commute and so in the limit $\tau \to 0$ the combined rotation simplifies to:

$$U(2\tau) = \mathrm{e}^{-i(h\tau S_z + h\tau S_y)} = \mathrm{e}^{-i\sqrt{2}h\tau\frac{(0,1,1)}{\sqrt{2}}\cdot\mathbf{S}} = \mathrm{e}^{-i\phi_0\mathbf{n_0}\cdot\mathbf{S}}$$

And so it follows that $\phi_0 = \sqrt{2}h \to \Omega_0 = \frac{1}{\sqrt{2}}h$.

(4) The rotation axis is also found from the expression above:

$$\hat{\mathbf{n}}_{\mathbf{0}} = \frac{1}{\sqrt{2}}(0,1,1)$$

(5) The tangent of the angle between two unit vectors is given by the ratio between their cross product and dot product.

$$tan(\theta) = \frac{\mathbf{n_0} \times \mathbf{n}}{\mathbf{n_0}\cdot\mathbf{n}} = \frac{1}{\sqrt{2}}\tan\left(\frac{h\tau}{2}\right)$$