E452: Operating with rotation matrices on spin states (2003A3)

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The problem:

Consider a spin $\frac{1}{2}$ state: $|\psi\rangle = \left(\mathrm{e}^{-i\frac{\pi}{4}}\cos\frac{\pi}{8}\right)|\uparrow\rangle + \left(\sin\frac{\pi}{8}\right)|\downarrow\rangle$.

(1) What is the spacial orientation of the state represented by $|\psi\rangle$?
(2) Write the rotation matrices needed to rotate a spin up $|\uparrow\rangle$ state to the $|\psi\rangle$ state.

Now consider a spin 1 up state (circular z polarization)

(3) Which state do we get after rotating it as in (2)?
(4) What is the probability to measure a linear z polarization after the rotation done in (3)?

The solution:

(1) The general spin $\frac{1}{2}$ polarization state, in the direction θ, φ is given by:

$$\begin{pmatrix} \mathrm{e}^{-i\varphi/2}\cos\left(\frac{\theta}{2}\right) \\ \mathrm{e}^{i\varphi/2}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

After multiplying it by the (immaterial) phase $\mathrm{e}^{-i\frac{\varphi}{2}}$, we get:

$$\begin{pmatrix} \mathrm{e}^{-i\varphi}\cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

Now it is evident, that in our state $|\psi\rangle$: $\theta = \varphi = \frac{\pi}{4}$

(2) The rotation written explicitly is:

$$|\psi\rangle = \mathrm{e}^{-i\varphi/2}R_z(\varphi)R_y(\theta)|\uparrow\rangle = \mathrm{e}^{-i\pi/8}\mathrm{e}^{-i\varphi S_z}\mathrm{e}^{-i\theta S_y}|\uparrow\rangle$$

The two rotation operators are represented by:

$$\mathrm{e}^{-i\frac{\pi}{4}\cdot\frac{1}{2}\sigma_y} = \cos\frac{\pi}{8} - i\sigma_y\sin\frac{\pi}{8} = \begin{pmatrix} \cos\pi/8 & -\sin\pi/8 \\ \sin\pi/8 & \cos\pi/8 \end{pmatrix}$$

$$\mathrm{e}^{-i\frac{\pi}{4}\cdot\frac{1}{2}\sigma_z} = \cos\frac{\pi}{8} - i\sigma_z\sin\frac{\pi}{8} = \begin{pmatrix} \mathrm{e}^{-i\pi/8} & 0 \\ 0 & \mathrm{e}^{i\pi/8} \end{pmatrix}$$

(3) Given the state: $|\Uparrow\rangle \longmapsto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, we will rotate it to get some other state $|\varphi\rangle$:

$$|\varphi\rangle = \mathrm{e}^{-i\frac{\pi}{4}S_z}\mathrm{e}^{-i\frac{\pi}{4}S_y}|\Uparrow\rangle$$

Using $\mathrm{e}^{-i\vec{\Phi}\cdot\vec{S}} = 1 - i\sin(\Phi)S_n - (1-\cos\Phi)S_n^2$ we obtain:

$$|\varphi\rangle = \begin{pmatrix} e^{-i\pi/4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\pi/4} \end{pmatrix} \begin{pmatrix} \frac{1}{2}\left(1+\cos\frac{\pi}{4}\right) & -\frac{1}{\sqrt{2}}\sin\frac{\pi}{4} & \frac{1}{2}\left(1-\cos\frac{\pi}{4}\right) \\ \frac{1}{\sqrt{2}}\sin\frac{\pi}{4} & \cos\frac{\pi}{4} & -\frac{1}{\sqrt{2}}\sin\frac{\pi}{4} \\ \frac{1}{2}\left(1-\cos\frac{\pi}{4}\right) & \frac{1}{\sqrt{2}}\sin\frac{\pi}{4} & \frac{1}{2}\left(1+\cos\frac{\pi}{4}\right) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Obviously, we need to use only the first column of the R_y matrix. After a little algebra we get:

$$|\varphi\rangle = \begin{pmatrix} \frac{\sqrt{2}+1}{4}(1-i) \\ \frac{1}{2} \\ \frac{\sqrt{2}-1}{4}(1+i) \end{pmatrix}$$

(4) First, we check and see that this ket is normalized. Then, we can find the probability:

$$|\langle \Updownarrow |\varphi\rangle|^2 = \frac{1}{4}$$