E4510: Operating rotation matrices on spin states (2002A2)

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The problem:

Find the state vector Ψ_m in the standard base for a spin polarized in the XY plane at angle $\varphi = 60°$. Relate to the following cases:

(1) Spin 1/2.
(2) Spin 1 with circular polarization.
(3) Spin 1 with linear polarization.

You must express the elements of the state vector in your final answer with $e^{i\pi/integer}$, $\sqrt{2}$ etc.
In order to prevent losing all points because of an algebraic mistake, please write first the analytic formula and only after calculate it to receive your final answer.

The solution:

(1) Any spin 1/2 polarization can be obtained by combining a rotation round the Y axis and a rotation round the Z axis on the "up" state:

$$|\vec{e}_{\theta,\varphi}>= R_z(\varphi)R_y(\theta)|\uparrow>= e^{-i\varphi S_z}e^{-i\theta S_y}|\uparrow>\rightarrow \begin{pmatrix} e^{-i\varphi/2}cos(\theta/2) \\ e^{i\varphi/2}sin(\theta/2) \end{pmatrix}$$

In order to get the question's polarization one needs to rotate by $\theta = \pi/2$ round the Y axis and then rotate by $\varphi = \pi/3$ round the Z axis. By doing so, one gets:

$$\Psi_m = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i(\pi/6)} \\ e^{i(\pi/6)} \end{pmatrix}$$

(2) Any right circular state can be obtained by combining a rotation round the Y axis and a rotation round the Z axis on the $|\Uparrow>$ state :

$$|\vec{e}_{\theta,\varphi}>= U(\varphi\vec{e}_z)U(\theta\vec{e}_y)|\Uparrow>= e^{-i\varphi S_z}e^{-i\theta S_y}|\Uparrow>\rightarrow \begin{pmatrix} \frac{1}{2}(1+cos\theta)e^{-i\varphi} \\ \frac{1}{\sqrt{2}}sin\theta \\ \frac{1}{2}(1-cos\theta)e^{i\varphi} \end{pmatrix}$$

The rotation action that needs to be done is a rotation by $\theta = \pi/2$ round the Y axis and then a rotation by $\varphi = \pi/3$ round the Z axis. Therefore:

$$\Psi_m = \frac{1}{2} \begin{pmatrix} e^{-i\pi/3} \\ \sqrt{2} \\ e^{i\pi/3} \end{pmatrix}$$

Any left circular state can be obtained by combining a rotation round the Y axis and a rotation round the Z axis on the $|\Downarrow>$ state :

$$|\vec{e}_{\theta,\varphi}>= U(\varphi\vec{e}_z)U(\theta\vec{e}_y)|\Downarrow>= e^{-i\varphi S_z}e^{-i\theta S_y}|\Downarrow>\rightarrow \begin{pmatrix} \frac{1}{2}(1-cos\theta)e^{-i\varphi} \\ -\frac{1}{\sqrt{2}}sin\theta \\ \frac{1}{2}(1+cos\theta)e^{i\varphi} \end{pmatrix}$$

The rotation action that needs to be done is a rotation by $\theta = \pi/2$ round the Y axis and after that a rotation by $\varphi = \pi/3$ round the Z axis. Therefore:

$$\Psi_m = \frac{1}{2}\begin{pmatrix} e^{-i\pi/3} \\ -\sqrt{2} \\ e^{i\pi/3} \end{pmatrix}$$

(3)Using the same technique as in (2) this time with a rotation matrix that rotates the state $|\Updownarrow>$ in order to get other linearly polarized states:

$$|\vec{e}_{\theta,\varphi}>= U(\varphi\vec{e}_z)U(\theta\vec{e}_y)|\Updownarrow>= e^{-i\varphi S_z}e^{-i\theta S_y}|\Updownarrow>\rightarrow \begin{pmatrix} -\frac{1}{\sqrt{2}}sin\theta e^{-i\varphi} \\ cos\theta \\ \frac{1}{\sqrt{2}}sin\theta e^{i\varphi} \end{pmatrix}$$

by using the same rotation as before $\theta = \pi/2$ and $\varphi = \pi/3$:

$$\Psi_m = \frac{1}{\sqrt{2}}\begin{pmatrix} -e^{-i\pi/3} \\ 0 \\ e^{i\pi/3} \end{pmatrix}$$