E4510: Operating rotation matrix on spin states

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The problem:

Find the state vector Ψ_m in the standard representation basis for spin polarized in the $x-y$ plain, at angle $\varphi = 60^o$. In your answer consider the following cases:

(1) Spin half.
(2) Spin one with circular polarization.
(3) Spin one with linear polarization.

In the final answer, state elements should be formulated using $e^{\frac{i\pi}{integer}}$, $\sqrt{2}$ and analogous.

The solution:

(1) Without loss of generality, we can generate any spin polarization state from the "up" state $|\uparrow\rangle \longrightarrow \begin{pmatrix}1\\0\end{pmatrix}$ by combining rotation around the y axis followed by a rotation around the z axis. Therefore, we can write:

$$|\Psi_m\rangle \longrightarrow R_z(\varphi)R_y(\theta)\begin{pmatrix}1\\0\end{pmatrix} = e^{-i\varphi S_z}e^{-i\theta S_y}\begin{pmatrix}1\\0\end{pmatrix} \tag{1}$$

Where S_y and S_z are the generators of rotations around the axis y and z respectively.
The general formula for constructing a 2x2 rotation matrix is:

$$R_n(\Phi) = \cos(\Phi/2)\hat{\mathbf{1}} - i\sin(\Phi/2)\sigma_n \tag{2}$$

Where $\sigma_n \equiv \vec{n}\cdot\vec{\sigma}$ by definition and $\hat{\mathbf{1}}$ is the identity matrix. Let us derive an expression for equation (1) above:

$$R_y(\theta) = \cos(\theta/2)\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} - i\sin(\theta/2)\begin{pmatrix}0 & -i\\i & 0\end{pmatrix} = \begin{pmatrix}\cos(\theta/2) & -\sin(\theta/2)\\\sin(\theta/2) & \cos(\theta/2)\end{pmatrix} \tag{3}$$

$$R_z(\varphi) = \cos(\varphi/2)\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} - i\sin(\varphi/2)\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} = \begin{pmatrix}e^{-i\varphi/2} & 0\\0 & e^{i\varphi/2}\end{pmatrix} \tag{4}$$

$$|\Psi_m\rangle = R_z(\varphi)R_y(\theta)\,|\uparrow\rangle \rightarrow \begin{pmatrix}e^{-i\varphi/2}\cos(\theta/2) & -e^{-i\varphi/2}\sin(\theta/2)\\e^{i\varphi/2}\sin(\theta/2) & e^{i\varphi/2}\cos(\theta/2)\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}e^{-i\varphi/2}\cos(\theta/2)\\e^{i\varphi/2}\sin(\theta/2)\end{pmatrix} \tag{5}$$

By submitting $\theta = \frac{\pi}{2}$ and $\varphi = \frac{\pi}{3}$ we rotate the "up" state into the $x-y$ plain and then rotate again to obtain the desired angle. The final result is therefore:

$$\Psi_m = \begin{pmatrix} e^{-i\frac{\pi}{6}}\cos(\frac{\pi}{4}) \\ e^{i\frac{\pi}{6}}\sin(\frac{\pi}{4}) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\frac{\pi}{6}} \\ e^{i\frac{\pi}{6}} \end{pmatrix} \tag{6}$$

(2) The procedure taken in the previous section is the same for the case of spin one, only this time the general formula for constructing a 3x3 rotation matrix is:

$$U_n(\Phi) = e^{-i\Phi S_n} = \hat{\mathbf{1}} - i\sin(\Phi)\cdot S_n - (\hat{\mathbf{1}} - \cos(\Phi))\cdot(S_n)^2 \tag{7}$$

Where $S_n \equiv \vec{n}\cdot\vec{S}$ by definition and $\hat{\mathbf{1}}$ is the identity matrix.

$$U_y(\theta) = e^{-i\theta S_y} = \begin{pmatrix} \frac{1}{2}(1+\cos(\theta)) & -\frac{1}{\sqrt{2}}\sin(\theta) & \frac{1}{2}(1-\cos(\theta)) \\ \frac{1}{\sqrt{2}}\sin(\theta) & \cos(\theta) & -\frac{1}{\sqrt{2}}\sin(\theta) \\ \frac{1}{2}(1-\cos(\theta)) & \frac{1}{\sqrt{2}}\sin(\theta) & \frac{1}{2}(1+\cos(\theta)) \end{pmatrix} \tag{8}$$

$$U_z(\varphi) = e^{-i\varphi S_z} = \begin{pmatrix} e^{-i\varphi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\varphi} \end{pmatrix} \tag{9}$$

Multiplying both matrixes would give us:

$$U_z(\varphi)U_y(\theta) = \begin{pmatrix} \frac{1}{2}e^{-i\varphi}(1+\cos(\theta)) & -\frac{1}{\sqrt{2}}e^{-i\varphi}\sin(\theta) & \frac{1}{2}e^{-i\varphi}(1-\cos(\theta)) \\ \frac{1}{\sqrt{2}}\sin(\theta) & \cos(\theta) & -\frac{1}{\sqrt{2}}\sin(\theta) \\ \frac{1}{2}e^{i\varphi}(1-\cos(\theta)) & \frac{1}{\sqrt{2}}e^{i\varphi}\sin(\theta) & \frac{1}{2}e^{i\varphi}(1+\cos(\theta)) \end{pmatrix} \tag{10}$$

So far we received the general expression for rotating spin one, but in order to get circularly polarized state we need to operate equation (10) on either one of the states represent circular polorization in the z direction . Without loss of generality we'll choose the state $|\Uparrow\rangle \longrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. We can now represent the requested state by:

$$|\Psi_m\rangle \longrightarrow U_z(\varphi)U_y(\theta)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{-i\varphi}(1+\cos(\theta)) \\ \sqrt{2}\sin(\theta) \\ e^{i\varphi}(1-\cos(\theta)) \end{pmatrix} \tag{11}$$

By submitting $\theta = \frac{\pi}{2}$ and $\varphi = \frac{\pi}{3}$ we get the final answer:

$$\Psi_m = \frac{1}{2}\begin{pmatrix} \mathrm{e}^{-i\frac{\pi}{3}} \\ \sqrt{2} \\ \mathrm{e}^{i\frac{\pi}{3}} \end{pmatrix} \tag{12}$$

It can be easily verified that if we had taken the state $|\Downarrow\rangle \longrightarrow \begin{pmatrix}0\\0\\1\end{pmatrix}$ we would have reached the same result only now the middle term would appear with a negative sign.

(3) Similarly, we can rotate the state $|\Updownarrow\rangle \longrightarrow \begin{pmatrix}0\\1\\0\end{pmatrix}$, which represent linear polariztion in the z direction, in order to get the requested linear polarization state:

$$\Psi_m = U_z(\varphi)U_y(\theta)\begin{pmatrix}0\\1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}-e^{-i\varphi}\sin(\theta)\\ \sqrt{2}\cos(\theta)\\ e^{i\varphi}\sin(\theta)\end{pmatrix} \tag{13}$$

By submitting $\theta = \frac{\pi}{2}$ and $\varphi = \frac{\pi}{3}$ we get the final answer:

$$\Psi_m = \frac{1}{\sqrt{2}}\begin{pmatrix}-\mathrm{e}^{-i\frac{\pi}{3}}\\ 0\\ \mathrm{e}^{i\frac{\pi}{3}}\end{pmatrix} \tag{14}$$

Notice that in each case above we started with normalized states ($\|\Psi\|^2 = 1$) and we received Ψ_m which is normalized as well. This suggests that the operation of rotation did not change the magnitude of the state, as expected.