Ex4484: Spin $\frac{1}{2}$ in a rotating magnetic field

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The problem:

The Hamiltonian of spin $\frac{1}{2}$ in a magnetic field is $H = h(t) \cdot S$.
Consider a magnetic field with magnitude $|h(t)| = \Omega_0$, which rotates in the XY-plane with angular velocity ω.
After time t the angle between X-axis and magnetic field is $\varphi = \omega \cdot t$.
The Hamiltonian of the "stationary system" (the lab) is $\tilde{H} = H - \omega \cdot S_z$.

1. Find the angle θ_0 in which one should prepare the spin in order for it to follow the magnetic field with a constant angle (θ_0 is respected to Z-axis)

   From now on, the spin is prepared (t=0) in X-axis direction

2. Letting the system to evolve, what is the maximal deviation angle of the spin from the X-axis

   From now on, we let the magnetic field to propagate for half a cycle ($\varphi = \pi$)

3. What is the final direction of the spin in the stationary system in the limit $\omega \mapsto 0$?
4. What is the final direction of the spin in the stationary system in the limit $\omega \mapsto \infty$?
5. For which ω values the deviation of the spin from X-axis would be 0?

Hint: this question does not require algebraic effort, only geometrical and physical understanding.

The solution:

1. In order for the spin to follow the magnetic field at a constant angle, it should be prepared in the direction of the magnetic field. By switching to a stationary system we get an effective magnetic field, so one must prepare the spin in the direction of the effective magnetic field.

   Magnetic field rotates in XY-plane:

$$\vec{\Omega} = \Omega_0(\cos\varphi, \sin\varphi, 0) \tag{1}$$

$$\vec{S} = (S_x, S_y, S_z) \tag{2}$$

   Hamiltonian of the stationary system would be:

$$\tilde{H} = \vec{\Omega}\cdot\vec{S} - \omega\cdot S_z = \frac{\Omega_0}{2}\cos\varphi\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + \frac{\Omega_0}{2}\sin\varphi\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} - \frac{\omega}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \tag{3}$$

$$= \frac{1}{2}\begin{bmatrix}-\omega & \Omega_0 e^{-i\varphi}\\ \Omega_0 e^{i\varphi} & \omega\end{bmatrix} \tag{4}$$

   Without the loss of generality one could assume that the effective magnetic field is displaced on XZ-plane, hence $\varphi = 0$:

$$\tilde{H} = \frac{1}{2}\begin{bmatrix}-\omega & \Omega_0\\ \Omega_0 & \omega\end{bmatrix} = \frac{1}{2}\left(\Omega_0\cdot\sigma_x - \omega\cdot\sigma_z\right) = \tilde{\Omega}\cdot\vec{S} \tag{5}$$

$$\tilde{\Omega} = (\Omega_0, 0, -\omega) \to \tilde{\theta}_0 = \arctan\left(\frac{\omega}{\Omega_0}\right) \tag{6}$$

   So the angle θ_0 would be:

$$\theta_0 = \frac{\pi}{2} + \tilde{\theta}_0 = \frac{\pi}{2} + \arctan\left(\frac{\omega}{\Omega_0}\right) \tag{7}$$

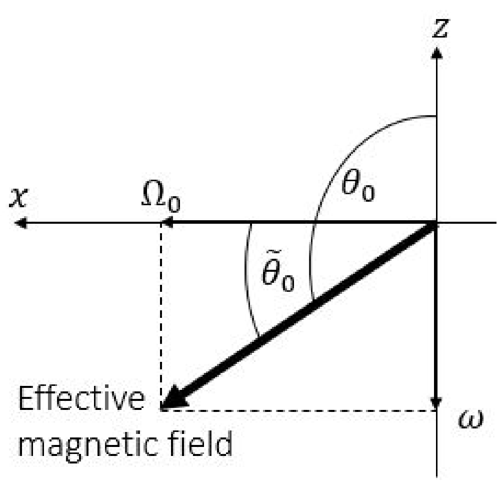

Figure 1: stationary system

2. Consider the effective magnetic field on XZ-plane with $\tilde{\theta}_0$ angle between the field and X-axis, by preparing the spin on X-axis, the spin would do a precession motion around the effective magnetic field, as a result the spin will be at the maximum angle deviation $\Delta\theta_{Max} = 2\tilde{\theta}_0$

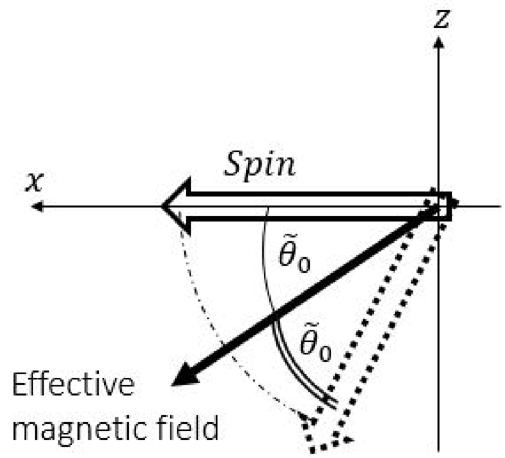

Figure 2: Spin precession in the stationary system

3. For the $\omega \mapsto 0$ limit, the change is adiabatic, hence the system will conserve its energy level, as a result the spin would follow the magnetic field, (continuing its percession motion) facing $\varphi = \pi$ respectfully to XY-plane.

4. For the $\omega \mapsto \infty$ limit, the change is sudden, so the state of the system before the change is conserved, hence the spin would stay in its initial state in $\varphi = 0$ respectfully to XY-plane.

5. Magnetic field rotates to $-\hat{x}$, finishing half of its period. Full period would be:

$$\omega \cdot T_m = 2\pi \rightarrow T_m = \frac{2\pi}{\omega} \tag{8}$$

For each half a period of the magnetic field the spin must finish a full period, by using this constrain on the spin's precession, and the fact that the spin precession frequency is Ω we get:

$$T_{Spin} = \frac{\pi}{\omega} \tag{9}$$

$$\Omega = \sqrt{{\Omega_0}^2 + \omega^2} \tag{10}$$

$$\Omega \cdot T_{Spin} = 2\pi \cdot n \tag{11}$$

$$\omega_n = \frac{\Omega_0}{\sqrt{4n^2 - 1}}, \quad n = 1, 2, 3, ... \tag{12}$$