Ex4482: Precession in a three sites system

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The problem:

A system is characterized by an energy spectrum E_n. The system is set up in a state ψ. Given the survival probability $p_n = |\langle En|\psi\rangle|^2$ which describes the initial state of the system.
$P(t)$ is the probablity find the system in the initial state after a time t.

$$\overline{P} = \lim_{t\to\infty} \frac{1}{t} \int_0^t P(t)$$

(0) Express $\overline{P}$ in terms of the survival probablity p_n.
A particle of charge q is in a three site system.
The standard basis is represented by the states $|x=1\rangle$,$|x=0\rangle$,$|x=-1\rangle$.

The Hamiltonian of the system is: $\begin{pmatrix} 0 & c & 0 \\ c & 0 & c \\ 0 & c & 0 \end{pmatrix}$

An external electric field f is inserted along the x-axis. The initial state of the particle is given by : $|x=1\rangle$.

(1) Express the Hamiltonian in terms of the generators of rotations S_x, S_y, S_z for a spin 1 particle.
(2) what is the fundamental frequency ω of the periodic motion obtained by the system?
(3) Explicitly represent $R(t) = \langle x\rangle_t$
(4) The probability P(t) can be expressed or written as a Fourier sum of the cosine sine functions, Write explicitly the expression of the Fourier sum of $P(t)$ of the non zero terms. There is no need to calculate the $A_n \& B_n$.
(5) Use the results obtained in clause (0) in order to get an explicit expression of $\overline{P}$.
(6) What is the minimal value $\overline{P}$ can obtain? For what electric field does this occur?

In clause (3) it is helpful to use the known results for precession (Rabi's Formula).
In clause (5) it is recommended to use the known polarization states of spin 1.

The solution:

(0) The initial state:

$$\psi(0) = \sum_n |E_n><E_n|\psi>$$

The general state:

$$\psi(t) = \sum_n e^{-iE_nt}|E_n><E_n|\psi>$$

So:

$$P(t) = |<\psi(0)|\psi(t)>|^2 = |\sum_n <E_n|\psi>^* <E_n| \sum_n <E_n|\psi> e^{-iE_nt}|E_n>|^2 =$$

$$|\sum_n < E_n|\psi >^* < E_n|\psi >< E_n|E_n > e^{-iE_nt}|^2 = |\sum_n | < E_n|\psi > |^2 < E_n|E_n > e^{-iE_nt}|^2$$

$$= \left|\sum_n p_n e^{-iE_nt}\right|^2$$

$$\overline{P} = \lim_{t\to\infty}\frac{1}{t}\int_0^t P(t)dt = \lim_{t\to\infty}\frac{1}{t}\int_0^t \left|\sum_n p_n e^{-iE_nt}\right|^2 dt = \lim_{t\to\infty}\frac{1}{t}\int_0^t \sum_{n,m} p_n p_m e^{-i(E_n-E_m)t}dt$$

For sum $n \neq m$ the Sine terms cancel each other

$$= \lim_{t\to\infty}\frac{1}{t}\int_0^t \sum_n p_n^2 + \sum_{n\neq m} 2cos((E_n - E_m)t)dt$$

$$\bar{P} = \lim_{t\to\infty}\frac{1}{t}[(\sum_n p_n^2)t + \frac{2sin((E_n - E_m)t)}{E_n - E_m}]$$

$$\bar{P} = \sum_n p_n^2$$

(1)The Hamiltonian is:

$$H = H_0 + V = \begin{pmatrix} 0 & c & 0 \\ c & 0 & c \\ 0 & c & 0 \end{pmatrix} - qfx$$

The intial state is $|x = 1\rangle$.

The matrix representation for the position operator x in the standard basis is:

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and for the generators of rotations:

$$S_x = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

It easy to see that :

$$H = \sqrt{2}cS_x - qfS_z$$

(2) The evolution operator is:

$$U(t) = e^{-it\hat{H}} = e^{-i(\overrightarrow{\Omega t})\overrightarrow{s}}$$

$$H = \overrightarrow{\Omega} \cdot \overrightarrow{S} = \Omega_x S_x + \Omega_y S_y + \Omega_z S_z$$

$$\overrightarrow{\Omega} = (\sqrt{2}c, 0, -qf)$$

$$|\overrightarrow{\Omega}| = \sqrt{(\sqrt{2}c)^2 + (qf)^2}$$

(3)

$$\langle x \rangle = \left\langle \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} \right\rangle$$

$$\langle x \rangle =< \psi(t)|x|\psi(t) >= P(x=1) * 1 + P(x=0) * 0 + P(x=-1) * -1$$

now its look like spin $\frac{1}{2}$ problem.

if we take a new S_x dim=2 with changing the 'c' to 2'c' as the energy gain, we preserve the eigenvalues. The evolution in time operator stay the same.
so

$$\theta = \arctan(\frac{\sqrt{2}c}{qf})$$

$$U(t)|\uparrow>= R(\overrightarrow{\theta}(t)) = cos(\frac{\Omega t}{2}) - isin(\frac{\Omega t}{2})[cos(\theta)\sigma_z + \sin(\theta)\sigma_x]$$

$$|\psi(t)>= U(t)|\uparrow>= [cos(\frac{\Omega t}{2}) - isin(\frac{\Omega t}{2})cos(\theta)]|\uparrow> -i[\sin(\theta)sin(\frac{\Omega t}{2}])|\downarrow>$$

$$\langle R(t) \rangle = \langle x \rangle =< \psi(t)|x|\psi(t) >=< \psi(t)|\{[cos(\frac{\Omega t}{2}) - isin(\frac{\Omega t}{2})cos(\theta)]|\uparrow> +i[\sin(\theta)sin(\frac{\Omega t}{2}])|\downarrow>\}$$

$$= cos(\frac{\Omega t}{2})^2 + sin(\frac{\Omega t}{2})^2 cos(\theta)^2 - \sin(\theta)^2 sin(\frac{\Omega t}{2})^2 = 1 - 2\sin(\theta)^2 sin(\frac{\Omega t}{2})^2$$

(4)The probability to find the system in the initial state after time t is:

$$P(t) = |\langle 1|\psi(t)\rangle|^2 = \sum_{n,m} p_n^* p_m e^{i(E_n - E_m)t} = \left\{ E_n - E_m = \tilde{\Omega} \right\}$$

we know that E_n get only values from $\{0, \Omega, -\Omega\}$ and $\tilde{\Omega}$ get $\{0, \Omega, -\Omega, 2\Omega, -2\Omega\}$:

For n=m $\Rightarrow A_0$

For n$\neq m \Rightarrow$ we have for $e^{i(E_n - E_m)t}$ 4 options(without duplications):

$$cos(\Omega t) + isin(\Omega t),\ cos(2\Omega t) + isin(2\Omega t), cos(-\Omega t) + isin(-\Omega t), cos(-2\Omega t) + isin(-2\Omega t)$$

The sine function cancelled and we stay only with:

$$A_0 + A_1 cos(\Omega t) + A_2 cos(2\Omega t)$$

(5)Explicit expression of $\overline{P}$ is:

Any rotation can be given by a combination of rotations around z axis and a rotation around the y axis. we find the state $|\uparrow>$ after rotations is circularly polarized. the circularly polarized state is the time average of the initial state rotation.

for $\theta = \arctan(\frac{\sqrt{2}c}{\epsilon})$

$$U(\varphi \overrightarrow{n_z})U(\theta \overrightarrow{n_y}) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(1 + cos\theta) \\ \frac{1}{\sqrt{2}} sin\theta \\ \frac{1}{2}(1 - cos\theta) \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

$$P_i = |\langle \psi_i | \psi \rangle|^2$$

$$\bar{P} = \sum_n p_n^2 = P^2(x = 1) + P^2(x = 0) + P^2(x = -1) =$$

$$\left(\frac{1}{2}(1 + cos\theta)\right)^4 + \left(\frac{1}{\sqrt{2}} sin\theta\right)^4 + \left(\frac{1}{2}(1 - cos\theta)\right)^4$$

(6)The minimal value $\overline{P}$ can obtain is:

$$\bar{P} = \tfrac{1}{16}\left[(1 + cos\theta)^4 + 4sin^4\theta + (1 - cos\theta)^4\right]$$

$$0 = \tfrac{\partial P}{\partial \theta} = \tfrac{1}{4}\left[-sin\theta (1 + cos\theta)^3 + 4cos\theta sin^3\theta + sin\theta (1 - cos\theta)^3\right]$$

$$0 = -(1 + cos\theta)^3 + 4cos\theta sin^2\theta + (1 - cos\theta)^3$$

$$0 = -\left(1 + cos^3\theta + 3cos\theta + 3cos^2\theta\right) + 4cos\theta sin^2\theta + \left(1 - cos^3\theta - 3cos\theta + 3cos^2\theta\right)$$

$$0 = 2cos\theta sin^2\theta - 4cos^3\theta - 6cos\theta$$

It can be true only when:

$$sin\theta = 0, cos\theta = 0$$

we get maximum for $sin\theta = 0$ and minimum for $cos\theta = 0$

$$\bar{P} = \tfrac{1}{16}\left[(1)^4 + 4 + (1)^4\right] = \tfrac{6}{16} = \tfrac{3}{8}$$

The required electric field is:

$$0 = cos\theta = \tfrac{qf_{min}}{\sqrt{2c^2 + q^2 f^2}}$$

$$f_{min} = 0$$