E444: Identification of Rotation

Submitted by: Uri Argaman and Yair Margalit

The problem:

Assume a particle with spin 1/2. The following rotation matrices are defined (in the standard basis):

$$\mathcal{R}_{\mathcal{A}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\mathcal{R}_{\mathcal{B}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\mathcal{R}_{\mathcal{C}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

(1) Find the angle of rotation Φ and the axis of rotation $\vec{n}$ for all of the matrices.
(2) Write a rotation matrix 4x4 that operates R_C on 2 spins.

The solution:
(1) A general rotation of spin $\frac{1}{2}$ is given by:

$$R(\Phi) = cos(\frac{\Phi}{2})I - isin(\frac{\Phi}{2})\sigma_n$$

$$\sigma_n = \vec{n} \cdot \vec{\sigma}$$

(A)For R_A: We can see that $R_A = \sigma_x$. So using gauge freedom, we will demand:

$$R(\Phi) = [cos(\frac{\Phi}{2})I - isin(\frac{\Phi}{2})\sigma_n]e^{i\alpha} = \sigma_x$$

If we use $\Phi = \pi$ and $\vec{n} = (1, 0, 0)$ we get:

$$R(\pi) = -ie^{i\alpha}\sigma_x$$

So for $\alpha = \frac{\pi}{2}$ we get the desired result.

(B)Similarly for R_B; We can see that:

$$R_B = \frac{1}{\sqrt{2}}(I - i\sigma_y)$$

We will choose $\Phi = \frac{\pi}{2}$ and $\vec{n} = (0, 1, 0)$:

$$R(\frac{\pi}{2}) = \frac{1}{\sqrt{2}}[I - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}] = R_B$$

(C)For R_C, we see that:

$$R_C = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z)$$

So we will choose $\Phi = \pi$ and $\vec{n} = \frac{1}{\sqrt{2}}(1,0,1)$. Using gauge freedom we will demand:

$$R(\pi) = -ie^{i\alpha}\frac{1}{\sqrt{2}}(\sigma_x + \sigma_z) = -\frac{i}{\sqrt{2}}e^{i\alpha}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

And if we take $\alpha = \frac{\pi}{2}$ we get the desired result.

(2) The 2 dimensional standard basis is: $|\uparrow>= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\downarrow>= \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. So we will take:

$$|\uparrow\uparrow>=|\uparrow>\otimes|\uparrow>= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad |\uparrow\downarrow>=|\uparrow>\otimes|\downarrow>= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|\downarrow\uparrow>=|\downarrow>\otimes|\uparrow>= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad |\downarrow\downarrow>=|\downarrow>\otimes|\downarrow>= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

And the matrix that rotates the two spins together:

$$R_C^{(4)} = I \otimes R_C^{(2)} + R_C^{(2)} \otimes I = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -2 \end{pmatrix}$$

We can easily see that the first matrix (before summation) only rotates the second spin, and the second matrix only rotates the first spin.