E443: Identification Hadamard rotation of spin 1/2

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The problem:

Given a particle of spin 1/2 let's define the next operation (in the standard basis):
$R = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$
(1) Operating R on the up state. What will be the polarization direction?
(2) Operating R on the down sate. What will be the polarization direction?
(3) What is the rotation angle represented by R?

It is possible to realize R at the lab by operating magnetic field for 1 sec. At this time the Hamiltonian is of the form of $H = h \cdot \sigma + c$.
(4) Decide the direction and the size of the magnetic field h.
(5) Decide the value of the constant c.

The solution:

(1) By operating R on the up state, we get:

$$R\left|\uparrow\right\rangle = \frac{1}{\sqrt{2}}(\left|\uparrow\right\rangle + \left|\downarrow\right\rangle)$$

(2) By operating R on the down state, we get:

$$R\left|\downarrow\right\rangle = \frac{1}{\sqrt{2}}(\left|\uparrow\right\rangle - \left|\downarrow\right\rangle)$$

(3)We can see the R matrix as the rotation matrix multiplying by a phase factor:
Thus we will compare them.

$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \frac{1}{\sqrt{2}}\left(\sigma_x + \sigma_z\right) = \frac{1}{\sqrt{2}}\left(1, 0, 1\right) \cdot \left(\sigma_x, \sigma_y, \sigma_z\right) = \vec{n} \cdot \vec{\sigma}$
$\vec{n} \cdot \vec{\sigma} = e^{i\alpha}\left[cos\left(\frac{\Phi}{2}\right)\hat{1} - isin\left(\frac{\Phi}{2}\right) \cdot \sigma_n\right]$

One can see from the equation

$cos\left(\frac{\Phi}{2}\right) = 0 \qquad \Rightarrow \Phi = \pi$

$e^{i\alpha} \cdot \left(-isin\left(\frac{\Phi}{2}\right)\right) = 1 \qquad \Rightarrow \alpha = \frac{\pi}{2}$

(4)+(5) After 1 sec the evolution operator is equal to R.
Thus we will compare them.

$\hat{U(t)} = e^{-i\hat{H}t} = e^{-i(h\cdot\sigma+c)t}$
$U(t = 1sec) = e^{-i\hat{h}\cdot\sigma} \cdot e^{-ic} = R = e^{i\alpha} \cdot e^{\frac{-i\hat{\Phi}\sigma}{2}}$

Derive from the comparison:
$c = -\alpha = -\frac{\pi}{2}$

$\hat{h} = |h| \cdot \hat{n}$

$|\hat{h}| = \frac{\pi}{2}, \hat{n} = \frac{1}{\sqrt{2}}(1, 0, 1)$