E442: Determining polarization state of spin $\frac{1}{2}$

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The problem:

Find the orientation of the polarized state

$$|\psi\rangle = \left(\mathrm{e}^{-i\frac{\pi}{4}} \cos\frac{\pi}{8}\right)|\uparrow\rangle + \left(\sin\frac{\pi}{8}\right)|\downarrow\rangle$$

using the two following methods:

(1) The first method is getting the polarized state by using the rotation matrices (around y axis and then around z axis) on the up $|\uparrow\rangle$ state.
(2) The second method is by calculating the polarization vector (i.e. the expectation values of σ_i)

The solution:

(1)first we find the polarization for a general state by rotating an up $|\uparrow\rangle$ state first by θ around y axis and then ϕ around z axis. It is given by:

$$|\vec{e_{\theta,\phi}}\rangle = R_z(\phi)R_y(\theta)|\uparrow\rangle = \mathrm{e}^{-i\phi S_z}\mathrm{e}^{-i\theta S_y}|\uparrow\rangle = \begin{pmatrix} \mathrm{e}^{-i\phi/2}\cos\left(\frac{\theta}{2}\right) \\ \mathrm{e}^{i\phi/2}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

Where

$$R_y(\theta) = \mathrm{e}^{-i\theta\cdot\frac{1}{2}\sigma_y} = \cos\frac{\theta}{2} - i\sigma_y \sin\frac{\theta}{2} = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$R_z(\phi) = \mathrm{e}^{-i\phi\cdot\frac{1}{2}\sigma_z} = \cos\frac{\phi}{2} - i\sigma_z \sin\frac{\phi}{2} = \begin{pmatrix} \mathrm{e}^{-i\frac{\phi}{2}} & 0 \\ 0 & \mathrm{e}^{i\frac{\phi}{2}} \end{pmatrix}$$

Thanks to gauge freedom, we can write the general state as:

$$|\vec{e_{\theta,\phi}}\rangle = \begin{pmatrix} \mathrm{e}^{-i\phi/2}\cos\left(\frac{\theta}{2}\right) \\ \mathrm{e}^{i\phi/2}\sin\left(\frac{\theta}{2}\right) \end{pmatrix} \cdot \mathrm{e}^{i\alpha}$$

So the state can be written with $\alpha = -\frac{\phi}{2} = -\frac{\pi}{8}$

$$|\vec{e_{\theta,\phi}}\rangle = \begin{pmatrix} \mathrm{e}^{-i\phi}\cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

Comparing $|\psi\rangle$ and $|\vec{e_{\theta,\phi}}\rangle$ gives $\theta = \frac{\pi}{4}$ and $\phi = \frac{\pi}{4}$

(2) The polarization vector is defined as:

$$\vec{M} = (\langle\sigma_x\rangle, \langle\sigma_y\rangle, \langle\sigma_z\rangle)$$

Where

$$\langle\sigma_i\rangle = \langle\Psi|\sigma_i|\Psi\rangle$$

For $|\psi\rangle$ state:

$$|\psi\rangle = \begin{pmatrix} e^{-i\frac{\pi}{4}} \cos\left(\frac{\pi}{8}\right) \\ \sin\left(\frac{\pi}{8}\right) \end{pmatrix}$$

$$\langle\psi|\sigma_x|\psi\rangle = \cos\left(\frac{\pi}{4}\right) \sin\left(\frac{\pi}{4}\right) = \frac{1}{2}$$

$$\langle\psi|\sigma_y|\psi\rangle = \sin\left(\frac{\pi}{4}\right) \sin\left(\frac{\pi}{4}\right) = \frac{1}{2}$$

$$\langle\psi|\sigma_z|\psi\rangle = |\psi_1|^2 - |\psi_2|^2 = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

Hence the polarization vector will be:

$$\vec{M} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right)$$

Therefore the spacial orientation will be: $\theta = \frac{\pi}{4}$ and $\phi = \frac{\pi}{4}$