E371: Electron(s) in Hall-Effect Geometry, (Moed A, 2004)

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The problem:

Assume a two dimensional rectangle model with length L and cyclic boundary condition on the interval $0 < x < L$, and width W $\left(-\frac{W}{2}\right) < y < \left(\frac{W}{2}\right)$
The specimen is located in a magnetic filed of the form :$\vec{B} = (0, 0, B(y))$ The specimen occupies electrons with no spin, mass m and charge e.

1. write the Hamiltonian of an electron in this geometry.

   assuming Landu gauge, ($p_x = \left(\frac{2\pi}{L}\right) l$) is constant of motion.

2. Write the effective potential $V_l(y)$ that determines the functions $\varphi_0(y)$

   Assume homogenous magntic filed $B(y) = B_0$ and that: ($\omega_B = \frac{eB_0}{m} > 0$) and also assume that only the lowest Landu level is occupied with electrons.

   Recall that each Landu level occupies a strip with width r_B and the distance between the strips is R_B

3. Write explicit expression for R_B, and, r_B, and for the condition $\frac{r_B}{R_B} << 1$. does the magntic filed need to be strong or weak in order to get a good separation between the levels?

   Tip: the ground state of the Harmonic oscillator is a Gaussian $\varphi_0(r) \propto \exp(-\frac{1}{2}m\Omega r^2)$

4. Assume that the width of the rectangular is $W = 3R_B$, so it contains only 3 electrons, write the wave function of each of the electrons, use the following notation: $|0\rangle, |1\rangle, |-1\rangle$

5. Suppose that the direction of the Magnetic filed changed only in the lower plain ($B(y < 0) = -B_0$) . Draw $V_l(y)$ for $l = 0, +1, -1$ Caution!

6. Write approximation for the wave function of each of the electrons, use the $|S0\rangle, |S\rangle, |A\rangle$ notation.

7. Guess what would be the energy splitting of the states $|S\rangle, |A\rangle$.

   Hint : find the "hight" of the barrier and the "overlap" factor of the gaussians in the superposition.

The Solution:

1. The Hamiltonian is : $H = \frac{1}{2m}(\vec{p} - \frac{e}{c}\vec{A})^2 + \hat{V}$

   using the "spirit" of Landau gage, we can define the vector potential as $\vec{A} = (-\int_0^y B(y')dy', 0, 0)$

   (It can be seen that when $B(y) = B_0$ we return to the normal Landau gage.) We also define the potential to be $\hat{V} = \hat{V_{Box}}$. and we get:

   $$\hat{H} = \frac{1}{2m}\left(\hat{p_x} + \frac{e}{c}\vec{A(y)}\right)^2 + \frac{1}{2m}\hat{p_y}^2 + \hat{V_{Box}}$$

   it's clear that $[\hat{H}, \hat{p_x}] = 0 => \hat{p_x}$ is a constant of motion, so we can define an effective potential:

   $$\hat{V}_{eff} = \frac{1}{2m}(\hat{p_x} + \frac{e}{c}\vec{A_x})^2 + \hat{V_{Box}}$$

2. and the Hamiltonian reduce to:

   $$\hat{H} = \frac{1}{2m}(\hat{p_x})^2 + \hat{V}_{eff}$$

3. If $B(y) = B_0$ then we return to the "Normal" Landau Gage $\vec{A} = (-B_o y, 0, 0)$ and the Hamiltonian becomes:
$\hat{H} = \frac{\hat{p}_y^2}{2M} + \frac{1}{2}M\omega_B^2(y - Y_l)^2$

where : $Y_l = \frac{1}{B_0}\frac{2\pi}{L_x}l$ and , $\omega_B = \frac{eB_0}{Mc}$
we can see that the Hamiltonian is of harmonic oscillation and a free particle, so using the separation of variable theorem, the wave function in the $|l, \nu\rangle$ base is of the form :

$$\psi = \frac{1}{\sqrt{L_x}} exp(-iB_0 Y_l x)\varphi_0(y - Y_l)$$

Therefore, we can see that the distance beteen the energy strips are quantumized by units of $\frac{1}{B_0}\frac{2\pi}{L_x}$, and we can conclude that the distance between each Landau state is exactly that:
$R_B = \frac{1}{B_0}\frac{2\pi}{L_x}$
The width of the strip the electrons occupies, is the width of the wave function, and in the ground state the wave function of harmonic oscillator is a Gaussian with width of :
$r_B = \frac{1}{\sqrt{M\omega_B}}$
we can now calculate the ratio:

$$\frac{r_B}{R_B} = \frac{\frac{1}{\sqrt{M\omega}}}{\frac{2\pi}{B_0 L_x}} = \frac{L}{2\pi}\sqrt{eB_0}$$

and if we demand that this ratio would be small then it follows that : $B_0 << \frac{4\pi^2 e}{cL_x^2}$

4. When we have only three electrons, we can write these states as follow :

$$|0\rangle = \frac{1}{\sqrt{L_x}}\varphi_0(y)$$

$$|+1\rangle = \frac{1}{\sqrt{L_x}} exp(\frac{i2\pi}{L_x}x)\varphi_0(y + \frac{2\pi}{B_0 L_x})$$

$$|-1\rangle = \frac{1}{\sqrt{L_x}} exp(-\frac{i2\pi}{L_x}x)\varphi_0(y - \frac{2\pi}{B_0 L_x})$$

5. Now we flip the dirction of $\vec{B}$ in the lower plain - this means that the vector potential changed to $\vec{A} = (B_0 y, 0, 0)$,in the lower part of the plain , and thus the Hamiltonyan is (in the lower region) $\hat{H}_{Lower} = \frac{\hat{p}_y^2}{2M} + \frac{1}{2}M\omega^2(y + Y_l)^2$
$\hat{H}_{Uper} = \frac{\hat{p}_y^2}{2M} + \frac{1}{2}M\omega^2(y - Y_l)^2$ where $Y_l = \frac{1}{B_0}\frac{2\pi}{L_x}l$ and , $\omega_B = \frac{eB_0}{Mc}$

so we can see tgat :

- when l = 0, then the potential is a parabola centred at the origin.
- when l is positive the potential is a parabola which is shifted, at the uper region to the right and in the lower region to the left. and since we demand that the wave function have to be continuous and smooth, then we get back a parabola center at the y = 0 but elevated a little.
- When l is negative then also we have 2 parbola shifted, but this time the uper side is shifted left and the lower side to the right, and the result is as illustrated below.

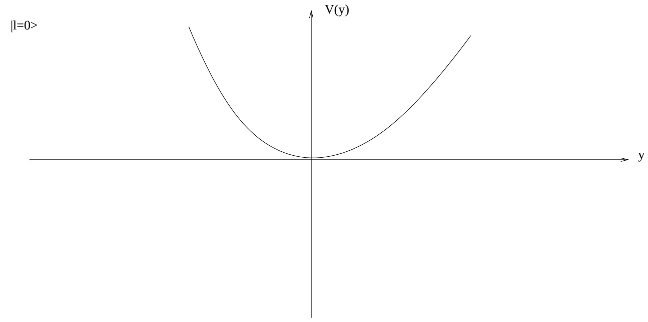

FIG. 1: l=0, parabola centred at the origin

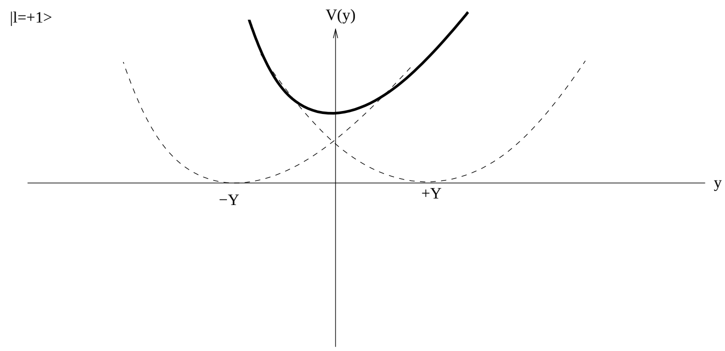

FIG. 2: l=+1,parabola elvated a litel

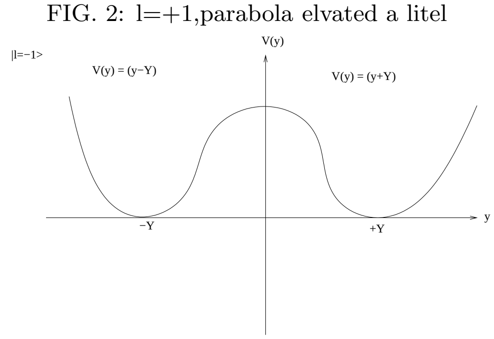

FIG. 3: l=-1, Two parabolas shfted

6. The energy state of the system:
   for l= 0, the problem reduce to one dimension potential and the ground state is $|S0\rangle = |0\rangle = \frac{1}{\sqrt{L_x}}\phi_0(y)$
   for $l = \pm 1$, the problem reduce to two sites problem, and the solutions are the symmetric and anti-symmetric wave function:

$$|S\rangle = \frac{1}{\sqrt{2}}\left(|+1\rangle + |-1\rangle\right) = \frac{1}{\sqrt{2L_x}} exp(-i\frac{2\pi}{L_x}x)\left(\varphi_0(y - R_B) + \varphi_0(y + R_B)\right)$$

$$|A\rangle = \frac{1}{\sqrt{2}}\left(|+1\rangle - |-1\rangle\right) = \frac{1}{\sqrt{2L_x}} exp(-i\frac{2\pi}{L_x}x)\left(\varphi_0(y - R_B) - \varphi_0(y + R_B)\right)$$

7. The energy splitting reflects the overlap of the two Gaussians $|+1\rangle$ and $|-1\rangle$ that were mixed into $|S\rangle$ and $|A\rangle$

$$EnergySplitting \propto exp(-(R_B/r_B)^2)$$