Question number 364

Submitted by: Yotam Soreq and Itamar Gurman

The question:
Theres given a two-dimensional potential pit $L \times L$, with a depth V_0. This means that inside the pit $V(x, y) = -V_0$, the walls of the pit are very high and $V(x, y) = 0$ outside the pit. In the consequence of this all electrons with energy $E > 0$ emerge out from the pit. Assume that remained electrons in the box have mass m, charge e and have no spin.
(1) Write a condition $L_{min} < L$ so that at least one electron could be located in the pit.
Assume below that $L >> L_{min}$.
(2) Calculate the number of electrons N_0 that could be located in the pit.
A homogenic magnetic field B is created perpendicular to the plane where the potential pit is stuated. In the next sections you can use known results about Landau levels (without any proof).
(3) Find the maximal field $B = B_2$ in order to have 2 Landau levels which are full in the hole.
We sign the number of particles in section 3 as N_2.
(4) Find $\frac{N_2}{N_0}$.

The solution:
(1)
It is known that for a given $L \times L$ infinite pit the energy levels are:

$$E_{n_x,n_y} = \frac{\pi^2}{2mL^2}\left(n_x^2 + n_y^2\right)$$

Cosidering the pit having very high walls on one side, and a depth of V_0 on the other side we get that the energy levels are:

$$E_{n_x,n_y} = \frac{\pi^2}{2mL^2}\left(n_x^2 + n_y^2\right) - V_0$$

In order to have at least one electron in the pit we have to look at the ground level, where $n_x = n_y = 1$. We also know that we need to demand $E_{1,1} < 0$ for the electron to stay in the pit. So we get:

$$E_{1,1} = \frac{\pi^2}{2mL^2}\left(1^2 + 1^2\right) - V_0 < 0$$

$$\Downarrow$$

$$L > \frac{\pi}{\sqrt{mV_0}}$$

Thus we get, $L_{min} = \frac{\pi}{\sqrt{mV_0}}$.
(2)
For $L >> L_{min}$ we know that the energy of the electrons that stay inside the pit is less than zero. Therefore we get the demand for every n_x and n_y :

$$n_x^2 + n_y^2 < \frac{2mL^2V_0}{\pi^2}$$

We can see that we recieved an equation that describes a circle with its middle in $(n_x = 0, n_y = 0)$ and a radius of $R^2 = \frac{2mL^2V_0}{\pi^2}$. Adding the knowledge that $n_x, n_y > 0$ and $L >> L_{min}$ we can say that N_0 is the area of a quarter circle with radius R.

$$N_0 = \frac{\pi R^2}{4} = \frac{mL^2V_0}{2\pi}$$

(3)
Let us choose Landau gauge $\vec{A} = (-By, 0, 0)$, and thus $[H, p_x] = 0$. The new Hamiltonian will be:

$$H = \frac{1}{2m}(p_x + By)^2 + \frac{p_y^2}{2m} - V_0$$

$$\Downarrow$$

$$H^{(n_x)} = \frac{1}{2m}\left(\frac{\pi n_x}{L} + By\right)^2 + \frac{p_y^2}{2m} - V_0$$

$$H^{(n_x)} = \frac{p_y^2}{2m} + \frac{B^2}{2m}\left(y + \frac{\pi n_x}{BL}\right)^2 - V_0$$

This Hamiltonian is of an harmonic oscillator in $\hat{y}$. Therefore its energy levels will be:

$$E_{n_y} = \frac{B}{m}\left(n_y + \frac{1}{2}\right) - V_0$$

In order for an electron to stay in the pit we demand $E_{n_y} < 0$. Two Landau levels mean $n_y = 1$, therefore:

$$E_{n_y=1} = \frac{B}{m}\left(1 + \frac{1}{2}\right) - V_0 < 0$$

$$\Downarrow$$

$$B < \frac{2}{3} m V_0 = B_2$$

(4)
Looking at the Hamiltonian we see that we have harmonic oscillators in $\hat{y}$. The distance between each one of them is $\frac{\pi}{BL}$. Since the size of the pit is L we can put $\frac{L}{\frac{\pi}{BL}}$ oscillators in it. This gives us the degeneration of the energy for each level:

$$N_2 = \frac{B_2 L^2}{\pi} = \frac{2mV_0 L^2}{3\pi}$$

Thus the ratio is:

$$\frac{N_2}{N_0} = \frac{\frac{2mV_0L^2}{3\pi}}{\frac{mL^2V_0}{2\pi}} = \frac{4}{3}$$