E361: Electron in the Hall effect geometry

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The problem:

Assume a two dimensional model with length L and cyclic boundary condition on the interval $0 < x < L$. The electric potential $V(y) = \frac{1}{2}\alpha y^2$ is given and a homogeneous magnetic field B at the z direction is applied. It convenient to express the magnetic field using the dimensionless parameter: $b = \frac{B}{\sqrt{m\alpha}}$. We can see that b is in fact the ratio between two frequency (What are they?) Inside the model we insert N spinless electrons.

(a) Show that as long as $N < [4(1+b^2)^3 m\alpha]^{1/4} L/\pi$ only the first landau band is populated.

(b) Show how you can get the result from (a) using very strong field or very weak field approximations.

The solution:

(a)

Let us look at the two frequencies of the system: ω_B - the frequency of circular motion due to Lorentz force, and ω_{osc} - the frequency of the oscillations due to the electric potential:

$$b = \frac{\omega_B}{\omega_{osc}} = \frac{(eB)}{(mc)}\sqrt{\frac{m}{\alpha}} = \frac{e}{c}\frac{B}{\sqrt{m\alpha}}$$

The Hamiltonaian that describes the motion of a particle in the Hall bar geometry is :

$$\hat{H} = \frac{1}{2m}(p_x + B\hat{y})^2 + \frac{1}{2m}p_y^2 + V(y)$$

We define the constant of motion:

$$Y = \hat{x} - \frac{1}{\omega_B}\hat{v}_y = -\frac{1}{B}\hat{p}_x$$

and the eigenvalues of $\hat{Y}$ are:

$$Y_l = \frac{1}{B}\frac{2\pi}{L}l$$

So, our Hamiltonian is:

$$H = \frac{1}{2m}p_y^2 + \frac{B^2}{2m}(y - Y_l)^2 + V(y).$$

We define the effective potential:

$$V^{(\ell)} = \frac{1}{2}m\omega_B^2(y - Y_l)^2 + V(y).$$

$$V^{(\ell)} = \frac{1}{2}m\omega_B^2(y^2 - 2yY_l + Y_l^2) + \frac{1}{2}\alpha y^2 = \frac{1}{2}m\omega_B^2(y^2 - 2yY_l + Y_l^2 + \frac{\alpha}{m\omega_B^2}y^2) = \frac{1}{2}m\omega_B^2\left[\left(1 + \frac{1}{b^2}\right)y^2 - 2yY_l + Y_l^2\right].$$

Now, lets write the equation in quadratic forms

$$\frac{1}{2}m\omega_B^2\left[\left(\sqrt{1+\frac{1}{b^2}}y - \sqrt{\frac{1}{1+1/b^2}}Y_l\right)^2 + Y_l^2\left(1 - \frac{1}{1+1/b^2}\right)\right] = \frac{1}{2}m\omega_B^2\left[\left(1+\frac{1}{b^2}\right)\left(y - \frac{1}{1+1/b^2}Y_l\right)^2 + Y_l^2\left(\frac{1}{b^2+1}\right)\right].$$

So, our Hamiltonian is

$$H = \frac{1}{2m}p_y^2 + \frac{1}{2}m\omega_B^2\left[\left(1+\frac{1}{b^2}\right)\left(y - \frac{1}{1+1/b^2}Y_l\right)^2 + Y_l^2\left(\frac{1}{b^2+1}\right)\right].$$

Let us define:

$$\tilde{y} = y - Y_l\frac{1}{1+1/b^2}$$

and

$$\tilde{\omega} = \omega_B\sqrt{1+\frac{1}{b^2}},$$

to get the new Hamiltonian

$$\tilde{H} = \frac{1}{2m}p_y^2 + \frac{1}{2}m\tilde{\omega}^2\tilde{y}^2 + \frac{1}{2}m\omega_B^2\frac{1}{b^2+1}Y_l^2.$$

The new eigenvalues are:

$$\tilde{E}_{l,\nu} = \tilde{\omega}(\frac{1}{2}+\nu) + \frac{1}{2}m\omega_B^2\frac{1}{b^2+1}Y_l^2.$$

The condition for filling the first Landau level is:

$$E_{l,0} < E_{0,1}$$

$$\frac{1}{2}m\omega_B^2\frac{1}{b^2+1}{Y_l}^2 < \tilde{\omega}$$

$$m\omega_B^2\frac{1}{b^2+1}\left(\frac{2\pi}{BL}\right)^2 l^2 < \tilde{\omega} = 2\omega_B\sqrt{1+\frac{1}{b^2}}$$

$$l^2 < 2\frac{1}{m\omega_B}\left(\frac{BL}{2\pi}\right)^2(b^2+1)\sqrt{1+\frac{1}{b^2}} = 2\frac{1}{m\omega_B}\left(\frac{BL}{2\pi}\right)^2\frac{(b^2+1)^{3/2}}{b}$$

$$l^2 < 2\frac{m}{mB}\left(\frac{BL}{2\pi}\right)^2(b^2+1)^{3/2}\frac{(m\alpha)^{1/2}}{B} = 2\left(\frac{L}{2\pi}\right)^2(b^2+1)^{3/2}(m\alpha)^{1/2}$$

$$|l| < [4(b^2+1)^3 m\alpha]^{1/4}\frac{L}{2\pi}$$

$$N = 2|l|$$

$$N < [4(b^2+1)^3 m\alpha]^{1/4}\frac{L}{\pi}$$

(b)

$$B << 1:$$

The Hamiltonaian is :

$$\hat{H} = \frac{1}{2m}p_x^2 + \frac{1}{2m}p_y^2 + V(y)$$

The eigenvalues are:

$$E_{l,\nu} = (\frac{1}{2} + \nu)\omega_{osc} + \frac{1}{2m}\left(\frac{2\pi l}{L}\right)^2$$

The condition for filling the first Landau level is:

$$\frac{1}{2m}\left(\frac{2\pi l}{L}\right)^2 < \omega_{osc}$$

$$l^2 < 2\sqrt{\frac{\alpha}{m}}\left(\frac{L}{2\pi}\right)^2 m$$

$$|l| < (\frac{1}{4}\alpha m)^{1/4}\frac{L}{\pi}$$

$$N < (4\alpha m)^{1/4}\frac{L}{\pi}$$

$$B >> 1:$$

For strong magnetic field with constant electric field we use the harmonic approximation then the energes are:

$$E_{l,\nu} \approx (\frac{1}{2} + \nu)\omega_B + V(Y_l) = (\frac{1}{2} + \nu)\omega_B + \frac{1}{2}m\omega_{osc}^2 Y_l^2$$

The condition for filling the first Landau level is:

$$\frac{1}{2}m\omega_{osc}^2 Y_l^2 < \omega_B$$

$$\frac{1}{2}\alpha\left(\frac{2\pi l}{BL}\right)^2 < \frac{B}{m}$$

$$|l| < \frac{BL}{2\pi}\left(\frac{2B}{m\alpha}\right)^{1/2} = \frac{L}{2\pi}\left(\frac{2B^3}{m\alpha}\right)^{1/2} = \frac{L}{2\pi}\left(\frac{2(\beta\sqrt{m\alpha})^3}{m\alpha}\right)^{1/2}$$

$$N < (4b^6 m\alpha)^{1/4}\frac{L}{\pi}$$