E3570: A particle on a disc with a homogeneous magnetic field - Landau levels

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The problem:

A particle is bound to move on the XY plane in the presence of a homogeneous magnetic field perpendicular to the plane.

(1) Write the Hamiltonian in Cartesian coordinates.
(2) Show that the Hamiltonian is of a two dimensional harmonic oscillator + a Zeeman term.
(3) Write down the Hamiltonian in polar coordinates and identify the radial equation.
(4) What are the energy levels determined by the radial equation without the Zeeman term? What is the degeneracy?
(5) What are the energy levels determined by the radial equation with the Zeeman term? What is the degeneracy?

Notes:

- In question 4 the answer can be deduced from the known solution in Cartesian coordinates.
- In question 5 assume that the particle is bound by a disc of radius R.
- Show that the results are in agreement with the known results for Landau levels.

The solution:

(1)

$$\vec{B} = B\hat{z} = \vec{\nabla} \times \vec{A}$$

It is convenient to use the symmetrical gauge:

$$\vec{A} = \left(-\frac{By}{2}, \frac{Bx}{2}, 0\right) = \frac{1}{2}\vec{B} \times \vec{r}$$

Therefore the Hamiltonian in Cartesian coordinates:

$$\mathcal{H} = \frac{\left(\vec{P} - \vec{A}\right)^2}{2M}$$

Using the triple multiplication associativity,

$$\vec{P} \cdot \vec{A} = \vec{A} \cdot \vec{P} = \frac{1}{2}\left(\vec{B} \times \vec{r}\right) \cdot \vec{P} = \frac{1}{2}\vec{B} \cdot \left(\vec{r} \times \vec{P}\right) = \frac{1}{2}\vec{B} \cdot \vec{L} = \frac{1}{2}BL_z$$

$$A^2 = \left(\frac{1}{2}\vec{B} \times \vec{r}\right) \cdot \left(\frac{1}{2}\vec{B} \times \vec{r}\right) = \frac{1}{4}\left(r^2B^2 - \left(\vec{r} \cdot \vec{B}\right)\right) = \frac{x^2 + y^2}{4}$$

Finally,

$$\mathcal{H} = \frac{P^2 + A^2 - BL_z}{2M} = \frac{P_x^2 + P_y^2}{2M} + \frac{B^2}{8M}\left(x^2 + y^2\right) - \frac{BL_z}{2M}$$

The second term is the diamagnetic term and the last is the Zeeman term.

(2) For a two dimensional harmonic oscillator we have

$$\mathcal{H}_{osc} = \frac{P^2}{2M} + \frac{1}{2}M\omega^2 r^2$$

taking $\omega = \frac{B}{2M}$ our Hamiltonian is

$$\mathcal{H} = \frac{P^2}{2M} + \frac{1}{2}M\omega^2 r^2 - \omega L_z$$

$$\mathcal{H} = \mathcal{H}_{osc} + \text{Zeeman}$$

(3)

$$-i\frac{\partial}{\partial \phi} = L_z \Rightarrow -\frac{\partial^2}{\partial \phi^2} = L_z^2$$

$$P = -i\nabla \Rightarrow P^2 = -\nabla^2$$

The Laplacian in polar coordinates is

$$\nabla^2 = \underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)}_{-P_r^2} + \frac{1}{r^2}\underbrace{\frac{\partial^2}{\partial \phi^2}}_{-L_z^2}$$

Hence, the Hamiltonian in polar coordinates is

$$\mathcal{H} = \frac{P_r^2}{2M} + \frac{L_z^2}{2Mr^2} + \frac{1}{2}M\omega^2 r^2 - \omega L_z$$

Working in a basis where L_z is diagonal, i.e

$$L_z|m\rangle = m|m\rangle \qquad m = 0, \pm 1, \pm 2, \dots$$

For each m,

$$\mathcal{H}^{(m)} = \frac{P_r^2}{2M} + \frac{m^2}{2Mr^2} + \frac{1}{2}M\omega^2 r^2 - \omega m$$

And the radial equation is:

$$\mathcal{H}^{(m)} f(r) = Ef(r)$$

$$\left(\frac{1}{2M}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{m^2}{2Mr^2} + \frac{1}{2}M\omega^2 r^2 - \omega m\right) f(r) = Ef(r)$$

(4) The energy levels of a 2-Dimensional harmonic oscillator are

$$E = \omega\left(n_x + n_y + 1\right)$$

And the degeneracy

$$g(E) = \frac{E}{\omega}$$

Since without the Zeeman term the Hamiltonians are identical, we require the same energy levels and degeneracies. In addition, we note that Hamiltonian contains only even functions of the quantum number m. Therefore we deduce that the energies must be

$$E = \omega(2\nu + |m| + 1)$$

$$\nu = 0, 1, 2, \ldots \qquad m = 0, \pm 1, \pm 2, \ldots$$

(5) With the Zeeman term, we get

$$E = \omega(2\nu + |m| + 1) - \omega m$$

$$= \omega(2\nu + |m| - m + 1)$$

Without being bounded by R, we would get an infinite degeneracy for positive m's.
For $\nu = 0$ only positive m's will result in the ground state energy.
Being bounded by a disc of radius R, m will be limited and so the degeneracy will be finite.
The effective radial potential is a function of m

$$V_{eff}^{(m)}(r) = \frac{m^2}{2Mr^2} + \frac{1}{2}M\omega^2 r^2$$

This potential has a minimum, r_{min}. The particle's wave function is centered around r_{min}, and is of a quantum ($\hbar$) scale. The disc has a macroscopic scale. Therefore we can neglect the edge effects and simply require that r_{min} be smaller than the disc's radius R.

$$\frac{d}{dr}V_{eff}^{(m)}(r) = -\frac{m^2}{Mr^3} + M\omega^2 r = 0$$

$$m = M\omega r_{min}^2 \qquad r_{min} < R$$

$$m < M\omega R^2$$

Remembering that $\omega = \frac{B}{2M}$ we get

$$m < \frac{BR^2}{2} = g$$

Where g is the degeneracy. which is exactly the known Landau degeneracy

$$g = \frac{L_x L_y}{2\pi} B$$

Since the area of disc $A = L_x L_y = \pi R^2$. Note that the above calculation is valid for $\nu = 0$. For $\nu > 0$ we get additional degeneracy due to the negative values of m. However, for a strong magnetic field or large R ($g \gg 1$) this is negligible.