E357: Particle bounded by harmonic potential + homogeneous magnetic field

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The problem:

Find the Landau levels and determine their dgeneracy using polar coordinates.

The solution:
Let us write the general Hamiltonian for a free particle in a homogeneous magnetic field.

$$\hat{H} = \frac{\hbar}{2M}(\vec{p} - \frac{e}{c}\vec{A})^2$$

It is convenience to gauge $\vec{A}$ in the following way in order to preserve the rotation symmetry:

$$\vec{A} = (-\frac{B}{2}y, \frac{B}{2}x, 0)$$

Now the Hamiltonian takes the form:

$$\hat{H} = \frac{{p_x}^2}{2M} + \frac{1}{2M}\left(\frac{eB}{2c}\right)^2 x^2 + \frac{{p_y}^2}{2M} + \frac{1}{2M}\left(\frac{eB}{2c}\right)^2 y^2 - \left(\frac{eB}{2cM}\right)\hat{L}_z$$

Let us define:

$$\Omega_0 \equiv \frac{eB}{cM}$$

$$\Rightarrow \hat{H} = \frac{{p_x}^2}{2M} + \frac{M}{2}\left(\frac{\Omega_0}{2}\right)^2 x^2 + \frac{{p_y}^2}{2M} + \frac{M}{2}\left(\frac{\Omega_0}{2}\right)^2 y^2 - \left(\frac{\Omega_0}{2}\right)\hat{L}_z$$

It is obvious to see that $[\hat{H}, \hat{L}_z] = 0$.

$$\hat{L}_z\psi = m\psi$$

while $m = 0, \pm1, \pm2....$
Beacouse the hamiltonian is commute whith $\hat{L}_z$ the hamiltonian can be expressed by block-diagonal.

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & -m \end{pmatrix}$$

From now on we will discuss only on the block that is not diagonal (that block is realated to the operator $\hat{x}^2, \hat{y}^2$).

We already know that for 2 dimension harmonic oscillator the energy levels are given by:

$$\Rightarrow E_{n_x,n_y} = \frac{\hbar\Omega_0}{2}(n_x + n_y + 1)$$

Will try to present the quantum numbers n_x, n_y (that are related to the Cartesian coordinated system) in terms of ν and m (that are related to the polar coordinated system).

$$n_x = n_x(\nu, m)$$

$$n_y = n_y(\nu, m)$$

$$n_x(\nu, m) + n_y(\nu, m) = f(\nu) + g(m)$$

$$\Rightarrow E_{\nu,m} = \frac{\hbar\Omega_0}{2}(f(\nu) + g(m) + 1)$$

We are looking for a transformation between the two sets of quantum numbers that will preserve the spectrum of the energy level.In particular:

- ground state
- the difference between the energy levels (ΔE).
- the number of dgeneracies in every energy level $\equiv N_g$

Meaning, the above values are invariant under the transformation.

Equal energy lines in the Cartesian plane are the lines that preserve the following equation:

$$n_x + n_y = const$$

We have the freedom to choose the $m = 0$ line as we wish.
A smart choice(In terms of symmetry) will be a choice that determines the line that perpendicular to the Equal energy lines and goes through the origin.
Will map the energy levels from the Cartesian plane to the polar plane, when the index m indicates the specific branch we are staying on and the index ν Indicates a specific level on that branch.
ν start from 0 because of the ground level.
Will start the mapping process:
First of all, will map the branch that associated with the ground level.

for $m = 1$:

- $E_{\nu=0,m=0} = \frac{\hbar\Omega_0}{2}$
- $\Delta E = \frac{\hbar\Omega_0}{2}$

$$E_{\nu,0} = E_{0,0} + \Delta E \cdot \nu = \frac{\hbar\Omega_0}{2}(2\nu + 0 + 1)$$

It is obvious to see that $f(\nu) = 2\nu$ and $g(m = 0) = 0$.

Now, for m=1:

- $E_{\nu=0,m=1} = 2\frac{\hbar\Omega_0}{2}$
- $\Delta E = \frac{\hbar\Omega_0}{2}$

$$E_{\nu,1} = \frac{\hbar\Omega_0}{2}(2\nu + 1 + 1)$$

It is obvious to see that $g(m = 1) = 1$.

in the same way, for m=-1, we obtain:

$$E_{\nu,-1} = \frac{\hbar\Omega_0}{2}(2\nu + 1 + 1)$$

and $g(m=-1)=1$.

and in general, for m=±m, we obtain:

- $E_{\nu=0,\pm m}=(m+1)2\frac{\hbar\Omega_0}{2}$
- $\Delta E=\frac{\hbar\Omega_0}{2}$

$$E_{\nu,\pm m}=\frac{\hbar\Omega_0}{2}(2\nu+m+1)$$

$$g(\pm m)=m$$

The only function that fulfill the above is g(m)=$|m|$.

For conclusion, we obtain the following expression:

$$E_{n_x,n_y}=\frac{\hbar\Omega_0}{2}(n_x+n_y+1)\Rightarrow E_{\nu,m}=\frac{\hbar\Omega_0}{2}(2\nu+|m|+1)$$

It is easy to see that N_g is conserved under the above transformation.

Finally, we obtain for the general hamiltonian: $E_{\nu,m}=\frac{\hbar\Omega_0}{2}(2\nu+|m|+1)-\frac{\hbar\Omega_0}{2}m$

$$\hat{H}=\frac{\hbar\Omega_0}{2}\begin{pmatrix}2\nu & 0 & 0\\ 0 & |m|+1 & 0\\ 0 & 0 & -m\end{pmatrix}$$

If we would have used Separation of variables in polar cordinates this is the hamiltonian that we would recieve.