E356: Particle bound by harmonic oscillator potential + Homogeneous magnetic field

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The problem:

A particle is bound by the two dimensional harmonic oscillator potential.

$$\mathrm{V}\left(\mathrm{r}\right)=\frac{1}{2}\mu\Omega_0^2 r^2$$

Ω_0 is the natural frequency of the Oscillator,μ is the mass of the Particle.

(1) Write the Energy levels.What is the degenerate of each level ?
The way to solve this problem is by separation of variables in cartesian coordinates. Next step is to translate this solution to a solution in polar coordinates.
(2) Write the Energy levels after adding the magnetic fields B.
(3) What is the generic degeneracy of each level ?
(4) What is the magnetic susceptibility if the system is in the ground state ?
(5) In our assumption was that the particle moves in two dimensions. What would be the answer in three dimension (the particle is free to move in the z directions),neglecting the diamagnetic term.

The solution:

(1) In cartesian coordinates the Hemiltonian of two dimensional oscillators is

$$H=\frac{p_x^2}{2\mu}+\frac{p_y^2}{2\mu}+\frac{1}{2}\mu\Omega_0^2\left(x^2+y^2\right)$$

and the energy levels are

$$E_{n_1,n_2}=\hbar\Omega_0\left(n_1+n_2+1\right)$$

$$n_1,n_2=0,1,2,3..$$

The degeneracy of the n level is

$$g\left(E_n\right)=n+1$$

we assume that physical system is independent of choosing a coordinate system then the energies and the degenerate in cartesian coordinates are the same as in polar coordinates.we are going to describe the following system in polar coordinates. In polar coordinates the Hemiltonian of two dimensional oscillators is

$$H_0=\frac{p_r^2}{2\mu}+\frac{L_z^2}{2\mu r^2}+\frac{1}{2}\mu\Omega_0^2 r^2$$

$$L_z=-i\hbar\frac{\partial}{\partial\phi}$$

The eigenvalue of L_z are $m=0\pm 1,\pm 2,\pm 3..$

$$p_r^2=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)$$

$$H^{(m)} = \frac{p_r^2}{2\mu} + \frac{m^2}{2\mu r^2} + \frac{1}{2}\mu\Omega_0^2 r^2$$

For each state with positive m there is an independent state with negative m with the same radial dependence and hence the same energy. So now letting m be any integer.The radial quantum number is ν it is non-negative integer. we translate this to a solution in polar coordinate system

$$E_{m,\nu} = \hbar\Omega_0 \left(2\nu + |m| + 1\right)$$

The ground state $(m = 0, \nu = 0)$ corresponding to $(n_1 = 0, n_2 = 0)$ is non-degenerate with energy $\hbar\omega$. Its $\frac{\hbar\Omega_0}{2}$ for every independent oscillator degree of freedom.The first excited state with energy $2\hbar\Omega_0$ is doubly degenerate,$(m = \pm 1, \nu = 0)$ as was the case previously,

(2) After adding the magnetic field B the Hmiltonian is

$$H = \frac{p^2}{2\mu} + \frac{1}{2}\mu\Omega_0^2(x^2 + y^2) + \frac{e^2B^2}{8\mu}(x^2 + y^2) - \frac{eB}{2\mu}L_z$$

can be written as $H = H_0 + H_1$, H_0 is Hmiltonianthe of Oscillator as in previous.H_1 belong to the external magnetic field . we defin cyclotron frequency to be Ω_B

$$\Omega_B = \frac{eB}{\mu}$$

And we get

$$H = \frac{p^2}{2\mu} + \frac{1}{2}\mu\Omega_0^2\left(x^2 + y^2\right) + \frac{1}{8}\mu\Omega_B^2\left(x^2 + y^2\right) - \frac{1}{2}\Omega_B L_z$$

we defin effective frequency to be Ω

$$\Omega^2 = \Omega_B^2 + \frac{1}{4}\Omega_0^2$$

$$H = \frac{p^2}{2\mu} + \frac{1}{2}\mu\Omega^2(x^2 + y^2) - \frac{1}{2}\Omega_B L_z$$

So the energies are of Particle in new two dimensional harmonic oscillator potential plus Zemman term $-\frac{1}{2}\Omega_B L_z$ In cartesian coordinates

$$E_{n_1,n_2} = \hbar\Omega\left(n_1 + n_2 + 1\right) - \frac{1}{2}\Omega_B L_z$$

and in polar coordinates

$$E_{m,\nu} = \hbar\Omega\left(2\nu + |m| + 1\right) - \frac{1}{2}\Omega_B m$$

(3) The energies levels are

$$E_{m,\nu} = \hbar\Omega\left(2\nu + |m| + 1\right) - \frac{1}{2}\Omega_B m$$

if β that is $\beta = \frac{\Omega}{\Omega_B}$ cannot be considered as a fraction than the generic degeneration of each level is zero. In the limit $\Omega_0 \to 0$ the energies levels are

$$E_{m,\nu} = \frac{1}{2}\hbar\Omega_B\left(2\nu + |m| - m + 1\right)$$

(4) The magnetic susiptability of the system in the ground stat $m = 0$,$\nu = 0$ is

$$M = -\frac{\partial E_{0,0}}{\partial B} = -\frac{\hbar B e^2}{8\mu^2}\left(\frac{B^2 e^2}{4\mu^2} + \Omega_0^2\right)^{-1/2}$$

$$\chi = \left.\frac{\partial M}{\partial B}\right|_{B=0} = -\frac{\hbar e^2}{8\mu^2 \Omega_0^2}$$

the susiptability less then zero.

(5) For a motion in three dimension in cartesian coordinates the Hemiltonian is

$$H = \frac{p_x^2}{2\mu} + \frac{p_y^2}{2\mu} + \frac{p_z^2}{2\mu} + \frac{1}{2}\mu\Omega_0^2\left(x^2 + y^2 + z^2\right)$$

and the energy levels are

$$E_{n_1,n_2,n_3} = \hbar\Omega_0\left(n_1 + n_2 + n_3 + \frac{3}{2}\right)$$

$$n_1, n_2, n_3 = 0, 1, 2, 3..$$

For the degree of degeneracy $n = n_1 + n_2 + n_3$ with $n_i = 0, 1, 2, ...$ For a given n we can choose a particular n_1. Then $n_2 + n_3 = n - n_1$. There are $n - n_1 + 1$ possible pairs n_2, n_3. n_2 can take on the values 0 to $n - 1$, and for each n_2 the value of n_3 is fixed. The degree of degeneracy therefore is

$$g\left(E_n\right) = \sum_{n_1=0}^{n}\left(n - n_1 + 1\right) = \sum_{n_1=0}^{n}\left(n+1\right) - \sum_{n_1=0}^{n}\left(n_1\right) = \left(n+1\right)\left(n+1\right) - \frac{n\left(n+1\right)}{2} = \frac{\left(n+1\right)\left(n+2\right)}{2}$$

In cylindrical coordinates the Hemiltonian is

$$H_0 = \frac{p_r^2}{2\mu} + \frac{L_z^2}{2\mu r^2} + \frac{1}{2}\mu\Omega_0^2 r^2 + \frac{p_z^2}{2\mu}$$

$$H^{(m)} = \frac{p_r^2}{2\mu} + \frac{m^2}{2\mu r^2} + \frac{1}{2}\mu\Omega_0^2 r^2 + \frac{p_z^2}{2\mu}$$

The energy levels are

$$E_{m,\nu,n} = \hbar\Omega_0\left(2\nu + |m| + n + \frac{3}{2}\right)$$

After adding the magnetic field B the Hmiltonian is

$$H = H_0 + H_1$$

$$H_1 = \frac{1}{8}m\Omega_B^2 r^2 - \frac{1}{2}\Omega_B L_z$$

And the energies levels are

$$E_{m,\nu,n} = \hbar\Omega\left(2\nu + |m| + 1\right) - \frac{1}{2}\Omega_B m + \hbar\Omega_0\left(n + \frac{1}{2}\right)$$

If we neglect the diamagnetic term the energy levels are

$$E_{m,\nu,n} = \hbar\Omega_0 \left(2\nu + |m| + n + \frac{3}{2}\right) - \frac{1}{2}\Omega_B m$$

if β that is $\beta = \frac{\Omega}{\Omega_B}$ cannot be considered as a fraction the generic degenerate of the energies is $2\nu + n$.

$$g(E_0) = g(E_1) = 1$$
$$g(E_2) = g(E_3) = 2$$

and in general

$$g(E_n) = \mathrm{int}\left(\frac{n}{2}\right) + 1$$

the magnetic susiptability of the system in the ground stat is as in two dimensions .