Ex3520: Hamiltonian of a particle in a non-uniform magnetic field

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The problem:

A particle of mass M, charge e ,spin 1/2 with gyromagnetic constant g is placed in a ring of radius R. A non-uniform magnetic field $B(r,\varphi,z) = br^{\alpha}$ is created in the z direction. In addition, there is a charge Q in the middle of the ring.

(1) Write an expression for the potential vector that is presented in the Hamiltonian of the particle.
(2) Write the Hamiltonian $H = h(L_z, S_z; M, e, g, R, b, \alpha, Q)$ with spin-orbit interaction.
(3) Write the effective gyromagnetic constant g_L as implied from the orbital Zeeman term in the Hamiltonian that you wrote.
(4) What should be the radius of the ring for the orbital Zeeman part will not affect the spectrum?

The solution:

(1) From Stokes' theorem we know that:

$$\Phi = \iint (rot\overline{A})d\overline{s} = \oint \overline{A}d\overline{l}$$

Applied for our problem, using symetric gauge:

$$\Phi = \int\int br^{\alpha}rdrd\varphi = AR\oint d\varphi$$

$$\Phi = 2\pi b \cdot \frac{R^{\alpha+2}}{\alpha+2} = 2\pi AR$$

$$\bar{A} = b \cdot \frac{R^{\alpha+1}}{\alpha+2}\hat{\varphi}$$

(2) The Hamiltonian of the particle is:

$$H = \frac{1}{2M}(\vec{p} - e\vec{A}(r))^2 + eV(r) - g\frac{e}{2M}B \cdot \vec{S} - (g-1)\frac{e}{2M^2}(\varepsilon \times \vec{p}) \cdot \vec{S}$$

Substituting into the Hamiltonian:$V(r) = \frac{Q}{R}$, $\varepsilon = \frac{Q}{R^2}\hat{r}$, $\vec{r} \times \vec{p} = \vec{L}$, $\vec{B}$ and $\vec{A}$, one gets:

$$H = \frac{1}{2MR^2}\hat{L}_z^2 - \frac{ebR^{\alpha}}{(2+\alpha)M}\hat{L}_z + \frac{e^2}{2M}(\frac{b}{\alpha+2})^2 R^{2(\alpha+1)} + \frac{eQ}{R} - g\frac{e}{2M}bR^{\alpha}\hat{S}_z - (g-1)\frac{eQ}{2M^2R^3}\hat{L}_z\hat{S}_z$$

(3) The Zeeman orbital term in the Hamiltonian of a particle in a homogeneous magnetic field is:

$$H_{\text{Zeeman,orbital motion}} = -\frac{e}{2M}B \cdot \vec{L}$$

Therefore we can find the effective gyromagnetic constant g_L:

$$-\frac{ebR^\alpha}{(2+\alpha)M}\hat{L_z} = -g_L\frac{e}{2M}B\cdot\vec{L}$$

$$g_L = \frac{2}{2+\alpha}$$

(4) From Aharonov-Bohm we know that the energies of a particle in a ring are:

$$E_n = \frac{1}{2MR^2}(n - \frac{e\Phi}{2\pi c})^2$$

if we swallow c into e, we get:

$$E_n = \frac{1}{2MR^2}(n - \frac{e\Phi}{2\pi})^2 = \frac{1}{2MR^2}(n - e\cdot b\cdot\frac{R^{\alpha+1}}{\alpha+2})^2$$

Thus, the magnetic field will not be noticeable by demanding:

$$eb\cdot\frac{R^{\alpha+2}}{\alpha+2} = integer$$

We can also treat the first term in the Hamiltonian of our problem:

$$\frac{1}{2M}(\vec{p} - e\vec{A}(r))^2$$

The eigenstates of this term are the momentum states $|k_n\rangle$ where:

$$k_n = \frac{2\pi n}{2\pi R} = \frac{n}{R} \quad , n = 0, \pm 1, \pm 2 ...$$

So the eigenvalues are:

$$E_n = \frac{1}{2MR^2}(n - e{\cdot}b\cdot\frac{R^{\alpha+1}}{\alpha+2})^2$$

Which is exactly the same expression we got from Aharonov-Bohm.