E3520: Hamiltonian of a particle in a non-uniform magnetic field

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The problem:

A particle of mass M and charge e with no spin is placed in a ring of radius R.
A non-uniform magnetic field $B(r, \varphi, z) = br^{\alpha}$ is created in the z direction.
Notice that the coordinate which is conjugated to the polar angle φ is the momentum L_z.

(1) Conclude what is the Zeeman term based on an analysis of the standard Hamiltonian.
(2) Write the Hamiltonian $H = h(L_z; M, e, R, b, \alpha)$ and the energy eigenvalues of the particle in the ring.
(3) Based on the previous part, what should be the radius of the ring for which the effect of the magnetic field on the spectrum will not be noticeable?
(4) Write a condition which allows answering the previous part without the need of finding the Hamiltonian.

The solution:

(1) We work with a cylindrical coordinate system (r, φ, z).
By using the gauge theory we set $\vec{A} = (0, A, 0)$.
From $\vec{B} = \vec{\nabla} \times \vec{A}$ we get the equation $br^{\alpha} = \frac{1}{r}\frac{\partial}{\partial r}(r \cdot A)$ which is solved for $A = \frac{b}{\alpha+2} \cdot r^{\alpha+1}$.

The standard Hamiltonian is:

$$H = \frac{1}{2M}(p - \frac{e}{c}A)^2$$

The Zeeman term is:

$$H_{Zeeman} = -\frac{1}{2M}\frac{e}{c} \cdot (Ap + pA) = -\frac{e}{2Mc} \cdot (2 \cdot p \cdot \frac{b}{\alpha+2} \cdot r^{\alpha+1}) = -\frac{eb}{Mc(\alpha+2)} r^{\alpha} L_z$$

(2) The Hamiltonian of the particle in the ring is:

$$H = \frac{1}{2M}(\frac{L_z}{R} - \frac{e}{c} \cdot \frac{b}{\alpha+2} R^{\alpha+1})^2 = \frac{L_z^2}{2MR^2} - \frac{eb}{Mc(\alpha+2)} R^{\alpha} L_z + \frac{e^2 b^2}{2Mc^2(\alpha+2)^2} R^{2(\alpha+1)}$$

The eigenstates of the Hamiltonian are $|l, m\rangle$ and the energy eigenvalues are:

$$E_{lm} = \frac{1}{2M}(\frac{m}{R} - \frac{e}{c} \cdot \frac{b}{\alpha+2} R^{\alpha+1})^2 = \frac{m^2}{2MR^2} - \frac{eb}{Mc(\alpha+2)} R^{\alpha} m + \frac{e^2 b^2}{2Mc^2(\alpha+2)^2} R^{2(\alpha+1)}$$

(3) Since m is an integer, by observing the first expression for the energies we realize that in order to get a spectrum of a particle in a ring without a magnetic field we have to demand:

$$\frac{eb}{c(\alpha+2)} R^{\alpha+2} = integer$$

By doing so, the energies are becoming $E_{lm} = \frac{1}{2MR^2} \cdot (integer)^2$. These are the energies of a particle in a ring without a magnetic field.

(4) From Aharonov-Bohm we know that the energies of a particle in a ring are:

$$E_n = \frac{1}{2MR^2}(n - \frac{e\Phi}{2\pi c})^2$$

Thus, the magnetic field will not be noticeable by demanding:

$$\frac{e\Phi}{2\pi c} = integer \Rightarrow \frac{e}{c}\Phi = 2\pi \cdot integer$$

The relation between the radius of the ring and the magnetic flux is easily calculated by using the formula $\Phi = \int\int \vec{B} \cdot \vec{da}$.