EX352: Hamiltonian in unform magnetic field

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The problem:

(1) A particle is placed in a uniform magnetic field. The field $\vec{B} = (0, 0, B)$.
Prove that the standart hamiltonian of the particle in the uniform magnetic field $H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)$
we can write as $H = \frac{p^2}{2m} - \frac{e}{2mc}BL_z + \frac{e^2B^2}{8mc^2}\left(x^2 + y^2\right)$.
(2)Derive using polar coordinate.
(3)What is the effective V(r) which is experienced by a particle with $L_z = m'$.
(4)Generalize your answers to the case of non-uniform magnetic field: $B = (0, 0, Br^a)$.

The solution:

(1) Some definitions: $\vec{p} = (p_x, p_y, p_z)$ and $L_z = \epsilon_{zij}r_ip_j = xp_y - yp_x$.
The pastulat $\vec{B} = rot\vec{A}$ is describing three differential equations.
By using the gauge transformation $A_z = 0$ we simplified the problem to one differential equation : $\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = B$,
so we get: $\vec{A} = \left(-\frac{1}{2}By, \frac{1}{2}Bx, 0\right)$.

$$H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 = \frac{1}{2m}\left\{\left(p_x^2 + \frac{e}{2c}By\right)^2 + \left(p_y^2 - \frac{e}{2c}Bx\right)^2 + p_z^2\right\} =$$

$$= \frac{1}{2m}\left\{\left(p_x^2 + p_y^2 + p_z^2\right) + \frac{eB}{c}\left(-xp_y + yp_x\right) + \left(\frac{eB}{2c}\right)^2\left(x^2 + y^2\right)\right\} =$$

$$= \frac{p^2}{2m} - \frac{e}{2mc}BL_z + \frac{e^2B^2}{8mc^2}\left(x^2 + y^2\right)$$

(2)To defined the Hamiltonian in cylindrical coordinates we shell use those relationships $x = rcos\varphi$, $y = rsin\varphi$
while L_z is not change when $(x, y, z) \Rightarrow (r, \varphi, z)$ and $p^2 = p_r^2 + \frac{L^2}{r^2}$ so we get:

$$H = \frac{1}{2m}\left(p_r^2 + \frac{L^2}{r^2}\right) - \frac{e}{2mc}BL_z + \frac{e^2B^2}{8mc^2}r^2$$

(3)The effective V(r) is:

$$V_{eff} = \frac{L^2}{2mr^2} - \frac{e}{2mc}BL_z + \frac{e^2B^2}{8mc^2}r^2$$

The effective V(r) which is experienced by a particle with $L_z = m'$ is:

$$V_{eff}|lm' >= \left(\frac{l(l+1)}{2mr^2} - \frac{eBm'}{2mc} + \frac{e^2B^2}{8mc^2}r^2\right)|lm' >$$

(4)By using the gauge transformation we set $A_r = A_z = 0$ in cylindrical coordinates from $\vec{B} = rot\vec{A}$ we get :

$$Br^a = \frac{1}{r}\frac{\partial\left(rA_\varphi\right)}{\partial r} \qquad \Rightarrow \qquad A_(r)\hat{\varphi} = \frac{B}{a+2}r^{a+1}$$

when

$$A_x = -A_\varphi sin\varphi = -\frac{yA_\varphi}{r} \qquad , \qquad A_y = A_\varphi cos\varphi = \frac{xA_\varphi}{r}$$

$$H = \frac{1}{2m}\left\{(p_x, p_y, p_z) - \frac{e}{c}\left(-\frac{yA_\varphi}{r}, \frac{xA_\varphi}{r}, 0\right)\right\}^2 =$$

$$= \frac{1}{2m}\left\{p^2 - \frac{2eB}{(a+2)c}r^a L_z + \left[\frac{eB}{(a+2)c}\right]^2 r^{2a+2}\right\}$$

It is easy to see that if $a = 0$ we have solution for particle in an uniform magnetic field. And more easy solution can be derived in polar coordinates:

$$H = \frac{1}{2m}\left\{(p', p_\varphi, p_z) - \frac{e}{c}\left(0, A(r), 0\right)\right\}^2 =$$

$$= \frac{1}{2m}\left\{p'^2 - (p', p_\varphi, p_z)\cdot\frac{e}{c}\left(0, A(r), 0\right) - \frac{e}{c}\left(0, A(r), 0\right)\cdot(p', p_\varphi, p_z) + \left(\frac{eA}{c}\right)^2\right\} =$$

$$= \frac{1}{2m}\left\{p'^2 - \frac{2e}{c}Ap_\varphi + \left(\frac{eA}{c}\right)^2\right\} =$$

$$= \frac{1}{2m}\left\{p_r^2 + \frac{L^2}{r^2} - \frac{2eB}{(a+2)c}r^a L_z + \left[\frac{eB}{(a+2)c}\right]^2 r^{2a+2}\right\}$$

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Remarks:

(2) is wrong.
(4) is better derived in polar coordinates.
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