Ex3070: Magnetization of a ring with electrons

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The problem:

A ring with radius R is placed in a uniform magnetic field B. In the ring there are 5 electrons with mass m, charge e and spin $\frac{1}{2}$.
The interaction term of the spin with the field is $g\frac{e}{2m}BS_Z$ where $S_z = \pm\frac{1}{2}$.
Note that two electrons can't be at the same state.
Let E be the ground state energy and $M(B) = -\frac{dE}{dB}$ be the magnetization.
B, R,m, e, and $g \approx 2.0023$ are given.
If the electrons are spin-less, then the magnetization is a periodic function of B with period B_0.
Later we will assume that $B = lB_0 + \triangle B$ where l is an integer and $0 < \triangle B << B_0$.

(1) Write an expression for B_0.
(2) Find $M(B)$ for $l = 0$ and $\triangle B \longrightarrow 0$.
(3) Write the addition to $M(B)$ for $l = 0$ and small $\triangle B$.
(4) Find $M(B)$ for large l and $\triangle B \longrightarrow 0$.
(5) Write the addition to $M(B)$ for large l and small $\triangle B$.
(6) Define large l.

Guidance:
in sections 2 and 4 it is recommended to draw an occupancy diagram of the energy levels.

The solution:

(1) The Hamiltonian of a spin-less electron in a ring of radius R inside a homogeneous magnetic field B is:

$$H = \frac{1}{2m}\left(P - \frac{e\Phi}{L}\right)^2$$

$$where: \Phi = \pi R^2 B, \qquad L = 2\pi R$$

The eigen-energies are:

$$E_n = \frac{1}{2mR^2}\left(n - \frac{eR^2B}{2}\right)^2$$

The ground state energy of five spin-less electrons E can be described in the the following graph:

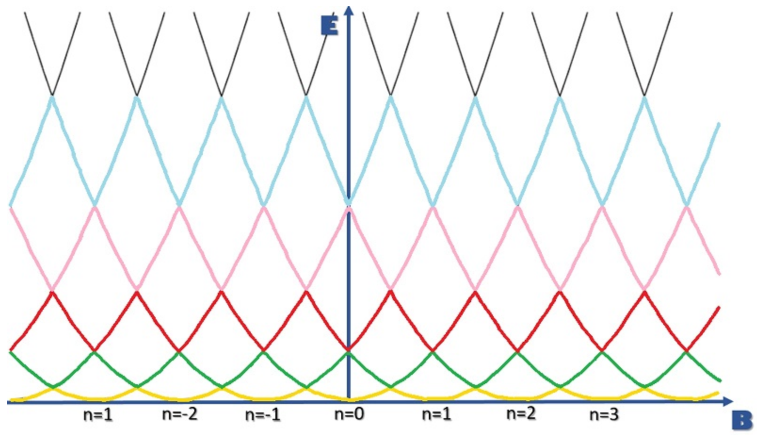

Figure 1: the ground state energy of five spin-less electrons

Each colored route describes the energy of one of the electrons as a function of B. For a given B the energy of the ground state is the sum of the energies of the five electrons for that specific B. From the graph one can see that the energy of each electron is periodic and the following equation exists:

$$E_i(n, B) = E_i(n+1, B+B_0) \qquad \Longrightarrow \qquad \frac{1}{2mR^2}\left(n-\frac{eR^2B}{2}\right)^2 = \frac{1}{2mR^2}\left(n+1-\frac{eR^2}{2}(B+B_0)\right)^2$$

Hence:

$$B_0 = \frac{2}{eR^2}$$

Accordingly, $M(B)$ which is received by the taking the derivative of the sum of those energies is also periodic with the same period B_0.

(2) If the electrons now have a spin, the $Spin-Zeeman$ term is added to the Hamiltonian and now the Hamiltonian is:

$$H \;=\; \frac{1}{2m}\left(P-\frac{e\Phi}{L}\right)^2 + \frac{ge}{2m}BS_Z$$

Accordingly, the eigen-energies are:

$$E_{n\pm} \;=\; \frac{1}{2mR^2}\left(n-\frac{eR^2B}{2}\right)^2 \pm \frac{ge}{4m}B$$

For small positive $B = \triangle B$ the electrons will be arranged in the following form:

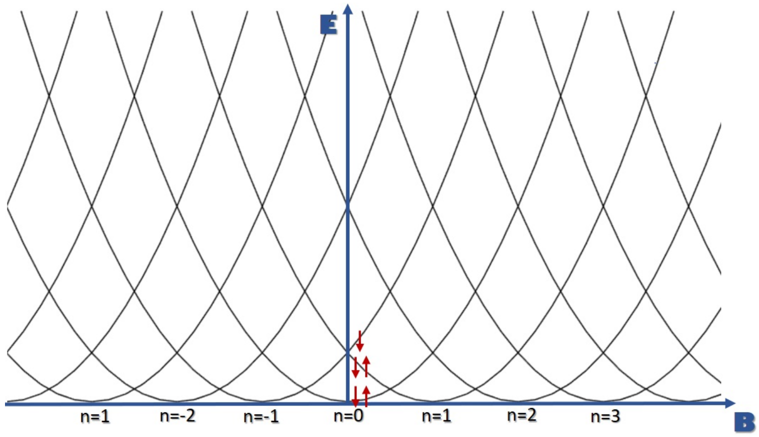

Figure 2: the arrangement in energy states of five electrons for small positive $B = \triangle B$

Therefore the ground state energy is:

$$E = \frac{5e}{4mB_0}\triangle B^2 - \frac{(2+g)e}{4m}\triangle B + \frac{3}{2mR^2}$$

And:

$$M(B) = -\frac{dE}{dB} = -\frac{5e}{2mB_0}\triangle B + \frac{(2+g)e}{4m}$$

For $\triangle B \longrightarrow 0$: $\qquad M(B) \longrightarrow \frac{(2+g)e}{4m}$

(3) The addition to $M(B)$ for small $\triangle B$ is:

$$-\frac{5e}{2mB_0}\triangle B$$

(4) For large B (large l) the main contribution to the eigen-energies is due to the $Spin - Zeeman$ term. Hence, all the 5 electrons will have spin down $|\downarrow\rangle$ and will arrange (for small positive $\triangle B$) in the following form:

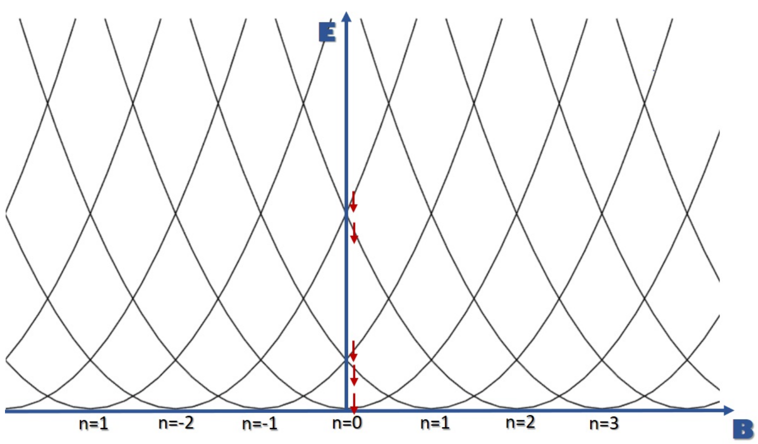

Figure 3: the arrangement in energy states of five electrons for large B

$M(B)$ in this case is similar to the case of spin-less electrons with the addition of a constant term. Hence $M(B)$ is independent of l and W.L.O.G we can choose $l = 0$ and then the ground state energy is:

$$E = \frac{1}{2mR^2}\left(5\left(\frac{\triangle B}{B_0}\right)^2 + 10\right) - \frac{5g}{2mR^2}\frac{\triangle B}{B_0}$$

And:

$$M(B) = -\frac{dE}{dB} = -\frac{5\triangle B}{mR^2 {B_0}^2} + \frac{5g}{2mB_0R^2}$$

So for $\triangle B \longrightarrow 0$: $\qquad M(B) \longrightarrow \frac{5g}{2mB_0R^2}$

(5) The addition to $M(B)$ for small $\triangle B$ is:

$$-\frac{5\triangle B}{mR^2 {B_0}^2}$$

(6) Large l is when the electron "prefers" to be in the fifth energy state with spin down $|{\downarrow}\rangle$ than in the lowest energy state with spin up $|{\uparrow}\rangle$.
For $\triangle B \longrightarrow 0$, $B \cong lB_0$ and $l = n$:

$$\frac{1}{2mR^2}\left(l - \frac{lB_0}{B_0}\right)^2 + \frac{ge}{4m}lB_0 > \frac{1}{2mR^2}\left(l - 2 - \frac{lB_0}{B_0}\right)^2 - \frac{ge}{4m}lB_0$$

So:

$$l > \frac{2}{g}$$