E3060: Magnetic Susceptibility of a Ring

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The problem:

Assume an electron with mass m, charge e and a gyromagnetic coefficient g, is confined to a one dimensional ring of length L. The ring is placed within a uniform magnetic field resulting in a flux of $\Phi = \left(L^2/4\pi\right) B$ through it. The system is prepared at zero temperature, that is, in its ground state. The magnetization and magnetic susceptibility are defined in terms of the first and second derivatives of the energy with respect to the magnetic field respectively, $M \equiv -\frac{\partial E}{\partial B}$ and $\chi \equiv -\frac{\partial^2 E}{\partial B^2}$. Remember that electrons are fermions with a spin of 1/2 which obey the Pauli Exclusion Principle.

(1) Find the magnetic susceptibility of the system for a single electron.
(2) Explain in what sense is it diamagnetic.
(3) What would the effect of a scattering barrier on the result you've obtained?
(4) Repeat the first question for 3 electrons.
(5) In the last case, what would the magnetization of the system be if $\lim B = 0$?
(6) How would your answer change if the flux was concentrated in the rings center?

The solution:

The Hamiltonian (Including the Zeeman-spin contribution) and the eigenenergies are

$$\mathcal{H} = \frac{1}{2\mathsf{m}}\left(p - \frac{e\Phi}{L}\right)^2 - g\frac{e_0}{2\mathsf{m}}\vec{B}\cdot\vec{S}$$

$$E_{n,\pm\frac{1}{2}} = \frac{1}{2\mathsf{m}}\left(\frac{2\pi}{L}n - \frac{eL}{4\pi}B\right)^2 \pm \left(\frac{ge}{4\mathsf{m}}\right)B$$

where $e_0 = |e|$.

(1) The magnetic susceptibility is

$$\chi = -\frac{\partial^2 E}{\partial B^2} = -\frac{1}{\mathsf{m}}\left(\frac{eL}{4\pi}\right)^2 = -\frac{e^2}{\mathsf{m}}\cdot\frac{\text{Area}}{4\pi}$$

(2) As χ is a negative constant, it creates a magnetic field opposing the external one and the energy levels rise as a quadratic function of B.

(3) The presence of the barrier will weaken the current thus reducing the diamagnetic effect. In the presence of an infinite barrier, that is a disconnected loop, there will be no current and no effect at all.

(4) As a result of their half spin and Pauli's exclusion principle, two of the electrons will have E_0 and another E_1. The susceptibility however as was shown in (1) doesn't depend on the occupied energy level leading to thrice the previous result,

$$E = 2E_0 + E_1 \Rightarrow \chi = 3\chi_{(1)} = -\frac{3}{\mathsf{m}}\left(\frac{eL}{4\pi}\right)^2$$

(5) The magnetization of an electron in the state $|n, s\rangle$ is given by

$$M_{n,\pm\frac{1}{2}} = -\frac{\partial E_n}{\partial B} = \frac{e}{4\mathsf{m}}\left(2n - eB\left(\frac{L}{2\pi}\right)^2 \mp g\right)$$

and as $B \to 0$, the magnetization of the system approaches

$$M_{0,+\frac{1}{2}} + M_{0,-\frac{1}{2}} + M_{\pm 1,\pm\frac{1}{2}} = \frac{e}{\mathsf{m}}\left(\pm\frac{1}{2} \mp \frac{g}{4}\right)$$

which considering that g is approximately 2 is either close to 0 or $\pm e/\mathsf{m}$. Assuming the electrons were setup in the lowest energy levels before the magnetic field was weakened $(B > 0)$, as a result of Zeeman splitting the higher energy electron will occupy the state $\left|-1, +\frac{1}{2}\right\rangle$, resulting in a magnetization of $-e/\mathsf{m}$, that is, aligned with the original fields direction.

(6) The difference would manifest in the Zeeman-spin contributions nullification as a result of having no magnetic field applied along their paths

$$M = -\frac{e}{2\mathsf{m}}$$

The magnitization here is a result of the non stationary electrons current and our choice of it's direction as to minimize the energy while the field was present $(B > 0)$.