Ex2660: Scattering matrix - The Mach-Zehnder Interferometer

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The problem:

The Mach-Zehnder Interferometer is built from two semi-reflecting mirrors and two regular mirrors. A regular mirror has a return amplitude of 1 so the only effect on the reflected beam is a phase change of π.There are a few passable ways to write the matrix of the semi-reflecting mirror here are two of them each describes a different mirror , the first symmetric and the second is not symmetric. (each matrix will gives a different result but we can build an interferometer from both)

A particles beam is split at the first semi-reflecting mirror to two beams which go through two different optical courses which cause a phase difference of ϕ.
They reunite in the second semi-reflecting mirror and the particles can exit in one of two channels.

1. Write a 4×4 matrix which describe a semi-reflecting mirror. The transmission (T) and reflection (R) are given. Explain why the semi-reflective mirror can be treated as a two channel junction problem with 2×2 matrix.
2. Write an expression for the two channels.
3. Make sure that the sum of the channels is one.

The solution:
original solution by Meny gabbay

(1)
We will use the lower left semi-reflecting mirror as our reference mirror and describe the 4 channels relative to it.
channel 1 is on the left (the light source)
channel 2 is the upper one.
channel 3 is on the right.
channel 4 is the lower one.
We will define the wave function in each channel as: $\psi_a = \frac{1}{\sqrt{v_a}}(A_a e^{-ikr} + B_a e^{+ikr})$ A is the amplitude of the incoming wave and B of the outgoing wave (from the mirror)
The S matrix is defined as: $B_b = \sum_a S_{ba} A_a$which is the matching conditions of the ingoing and outgoing amplitudes;
The properties of our S matrix:

1. Since the mirror isn't absorbing the light the intensity which goes in equal the intensity which goes out: $\sum |A_n|^2 = \sum |B_n|^2$ which mean $SS^\dagger = I$
2. Because of the symmetry, a beam coming from channel 1 and reflected to channel 2 is the same as a beam coming from channel 2 and reflected to channel 1 which mean: $S_{ij} = S_{ji}$
3. Because the mirror is slanted, a beam cannot be returned to the channel from which it came: $S_{ii} = 0$
4. Because the mirror is slanted, $S_{14} = S_{23} = S_{32} = S_{41} = 0$

Due to properties 2,3 and 4 the S matrix has only 8 unknowns (S_{ij} are complex numbers):

$$S = \begin{pmatrix} 0 & S_{12} & S_{13} & 0 \\ S_{12} & 0 & 0 & S_{24} \\ S_{13} & 0 & 0 & S_{34} \\ 0 & S_{24} & S_{34} & 0 \end{pmatrix}$$

Using property 1 we get the following equations:

$$\begin{aligned}
&|S_{12}|^2 + |S_{13}|^2 = 1 \\
&|S_{12}|^2 + |S_{24}|^2 = 1 \\
&|S_{13}|^2 + |S_{34}|^2 = 1 \\
&|S_{24}|^2 + |S_{34}|^2 = 1 \\
&S_{12}S_{24}^* + S_{13}S_{34}^* = 0 \\
&S_{12}S_{13}^* + S_{24}S_{34}^* = 0 \\
&S_{13}S_{12}^* + S_{34}S_{24}^* = 0 \\
&S_{24}S_{12}^* + S_{13}S_{34}^* = 0
\end{aligned}$$

The scattering matrix determine the matching conditions up to a phase so we can choose for example that $S_{13} = t$ and is a real number.
We use the rest of the equations above to determine the rest of the phases of each element of the matrix. after solving the above equation we get:

$$S = \begin{pmatrix} 0 & r & t & 0 \\ r & 0 & 0 & t \\ t & 0 & 0 & -r \\ 0 & t & -r & 0 \end{pmatrix}$$

$$R = |r|^2, T = |t|^2, R + T = 1$$

The semi-reflecting mirror can be treated as a two channel junction when the mirror is perpendicular to the beam making reflection to the same channel possible, and eliminating the channels parallel to the mirror.
(2)
The amplitude for each channel equals the sum over paths that went in it:

$$\begin{aligned}
\text{channel 2: } B_2 &= [t^2 e^{i\pi} e^{i\phi_1} - r^2 e^{i\pi} e^{i\phi_2}]A_0 \\
\text{channel 1: } B_1 &= [rte^{i\pi} e^{i\phi_1} + rte^{i\pi} e^{i\phi_2}]A_0
\end{aligned}$$

Where $\phi_1 = k * L_1$ and $\phi_2 = k * L_2$ and $A_0 = 1$ is the amplitude of the beam before it is split.
(3)
From here we get that:

$$|B_2|^2 = r^2t^2(2 + e^{i\phi} + e^{-i\phi}) = 2r^2t^2(1 + cos(\phi))$$

$$|B_1|^2 = r^4 + t^4 - r^2t^2e^{i\phi} - r^2t^2e^{-i\phi} = r^4 + t^4 - 2r^2t^2cos(\phi)$$

$$|B_1|^2 + |B_2|^2 = r^4 + t^4 + 2r^2t^2 = (r^2 + t^2)^2 = (R + T)^2 = 1$$

Were $\phi = \phi_2 - \phi_1$.