E262: Scattering matrix of I-junction

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The problem:

Three wires are attached in one point. Suppose that the signal coming from one of the legs is partly transmitted to other legs and is partly reflected back to the first leg. Define the scattering matrix as S which is 3×3

(1) Write S if given that all the reflection amplitudes are equal to real number r and all the transmission amplitudes are equal to positive real number t.
(2) Find the numerical values of r and t.
(3) Generalize the problem for a case of more than 3 wires.

The solution:

(1) The wave function can propogate in 3 directions. The scattering matrix is:

$$\mathcal{S} = \begin{pmatrix} r_1 & t_{21} & t_{31} \\ t_{12} & r_2 & t_{32} \\ t_{13} & t_{23} & r_3 \end{pmatrix}$$

We know that the system is symmetric, so we can say that $r1 = r2 = r3$ and also $t_{21} = t_{31} = t_{12} = t_{32} = t_{13} = t_{23}$. We obtain the following matrix:

$$\mathcal{S} = \begin{pmatrix} r & t & t \\ t & r & t \\ t & t & r \end{pmatrix}$$

(2)We know that t and r are real and also $t > 0$. Requiring that

$$S^{+} \cdot S = I$$

$$\begin{pmatrix} r & t & t \\ t & r & t \\ t & t & r \end{pmatrix} \cdot \begin{pmatrix} r & t & t \\ t & r & t \\ t & t & r \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we can obtain:

$$\begin{cases} r^2 + 2t^2 = 1 \\ rt + tr + t^2 = 0 \end{cases}$$

Solutions of these equations are:

$$r = \pm 1/3$$

$$t = \mp 2/3$$

Requiring $t > 0$ the solutions can be rewritten as:

$$r = -1/3$$

$$t = 2/3$$

(3)If we have more than 3 wires,we can write matrix as:

$$\mathcal{S} = \begin{pmatrix} r_1 & t_{12} & t_{13} & t_{14} & t_{15} & . & . & . \\ t_{21} & r_2 & t_{23} & t_{24} & t_{25} & . & . & . \\ t_{31} & t_{32} & r_3 & t_{34} & t_{35} & . & . & . \\ . & . & . & . & . & . & . & . \end{pmatrix}$$

We know that the system is symmetric, so we can say that $r_1 = r_2 = r_3 = r_4 = ... = r$ and also $t_{21} = t_{31} = t_{12} = t_{32} = t_{13} = t_{23} = ... = t$. We obtain the following matrix:

$$\mathcal{S} = \begin{pmatrix} r & t & t & t & t & . & . & . \\ t & r & t & t & t & . & . & . \\ t & t & r & t & t & . & . & . \\ . & . & . & . & . & . & . & . \end{pmatrix}$$

We know that t and r are real and also $t > 0$. Requiring that

$$S^{+} \cdot S = I$$

$$\begin{pmatrix} r & t & t & t & t & . & . & . \\ t & r & t & t & t & . & . & . \\ t & t & r & t & t & . & . & . \\ . & . & . & . & . & . & . & . \end{pmatrix} \cdot \begin{pmatrix} r & t & t & t & t & . & . & . \\ t & r & t & t & t & . & . & . \\ t & t & r & t & t & . & . & . \\ . & . & . & . & . & . & . & . \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & . & . \\ 0 & 1 & 0 & 0 & . & . \\ 0 & 0 & 1 & 0 & . & . \\ . & . & . & . & . & . \end{pmatrix}$$

we can obtain:

$$\begin{cases} r^2 + (n-1)t^2 = 1 \\ rt + tr + (n-2)t^2 = 0 \end{cases}$$

where n is number of the wires. Solutions of these equations are:

$$r = \pm(2/n) \mp 1$$

$$t = \pm 2/n$$

Requiring $t > 0$ the solutions can be rewritten as:

$$r = (2/n) - 1$$

$$t = 2/n$$