E255: finding the S matrix for a delta function in the origin:

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The problem:

For the potential $V(x) = u\delta(x)$ write the wave function as [sorry for the typo]

$$\Psi = Ae^{ikx} + Be^{-ikx}[x < 0]$$

$$\Psi = Ce^{ikx} + De^{-ikx}[x > 0]$$

(1) find the S matrix
(2) represent the S matrix with the transmission probability g and three phases.

The solution:

(1) Because the wave amplitude at all points must be defined we require:

$$\Psi^{+}(0) = \Psi^{-}(0)$$

$$A + B = C + D$$

And because we require integrability at all points we get:

$$\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}\frac{-1}{2m}\frac{d^2}{dx^2}\Psi + u\delta(x)\Psi dx = \int_{-\varepsilon}^{\varepsilon} i\frac{d}{dt}\Psi dx = 0$$

which, in a simpler writing would be:

$$\frac{1}{2m}[\frac{d}{dx}\Psi(\varepsilon) - \frac{d}{dx}\Psi(-\varepsilon)] = u\Psi(0)$$

which yields:

$$\frac{ik}{2m}C - \frac{ik}{2m}D - (\frac{ik}{2m}A - \frac{ik}{2m}B) = u(A + B) = u(C + D)$$

in terms of $u = \frac{U}{v_E}$ where $v_E = \sqrt{\frac{2E}{m}} = \frac{k}{m}$ we can simplify this expression:

$$C - D - (A - B) = -2ui(A + B)$$

Let us assume the wave comes from the left: A is then the coming wave amplitude,
B is the reflected wave amplitude, C is the transmitted wave amplitude,
and D does not exist: we get:
I.

$$A + B = C$$

and II.

$$C - A + B = -2iuC$$

now, for doing the same process again for a wave coming from the right, $A = 0$ for a wave coming from the right.
we get this set of equations:

III.

$$D + C = B$$

and IV.

$$C - D + B = -2iuB$$

in this case we get a matrix which looks like this:

$$\boldsymbol{S} = \begin{pmatrix} r & t \\ t & r \end{pmatrix}$$

where r is the relation between B and A or C and D (both are equal because the system is symetric around the Y axis), and in the same manner $t = \frac{B}{D} = \frac{C}{A}$.
by using the equation $(-A + B = C - 2A)$ on II we get:

$$2A = 2(iu + 1)C$$

$$t = \frac{1}{1 - iu}$$

we can see that if we subtitude C for $B - D$ on IV we get

$$2D = 2(iu + 1)B$$

which gives the same result for t.
if we substitute C for $A + B$ on II we get:

$$2B = -2iu(A + B)$$

from which we get the relation:

$$\frac{B}{A} = r = \frac{-iu}{1 + iu}$$

which is the same as $\frac{C}{D}$.
now we can write r and t:

$$t = \frac{1}{1 + iu} = \frac{1 - iu}{1 + u^2}$$

$$r = \frac{-iu}{1 + iu} = \frac{-u(i + u)}{1 + u^2}$$

so

$$\boldsymbol{S} = \begin{pmatrix} \frac{-iu}{1+iu} & \frac{1}{1+iu} \\ \frac{1}{1+iu} & \frac{-iu}{1+iu} \end{pmatrix}$$

(2)
the S matrix with the transmission probability g and three phases, would be written as such:

$$\mathbf{S} = \mathrm{e}^{i\gamma} \begin{pmatrix} i\sqrt{1-g}\mathrm{e}^{i\alpha} & \sqrt{g}\mathrm{e}^{-i\phi} \\ \sqrt{g}\mathrm{e}^{i\phi} & i\sqrt{1-g}\mathrm{e}^{-i\alpha} \end{pmatrix}$$

where

$$g = |t|^2 = \frac{1+u^2}{(1+u^2)^2} = \frac{1}{1+u^2}$$

we get the phase of i*t from:

$$\arctan \frac{\Im(it)}{\Re(it)} = \arctan \frac{1}{u}$$

and the same goes for r:

$$\arctan \frac{\Im(r)}{\Re(r)} = \arctan \frac{1}{u}$$

in both cases we get the phase $\arctan \frac{1}{u}$, so both $\sqrt{1-g}$ and $i\sqrt{g}$ have the same phase:

$$\gamma = \arctan \frac{1}{u}$$

because t and r for both waves coming from the left and from the right are the same (the problem is symetric), $\phi = \alpha = 0$. if this problem was not only one dimensional, we could have noticed the effect of the vector potential and get ϕ. and if the potential was a-symetric we would have gotten different r phases, and use α for it.

Summarizing formula the way I like [D.C.]:

The $\boldsymbol{S}$ matrix for a delta function scatterer $V(r) = u\delta(r)$ is

$$\boldsymbol{S} = \begin{pmatrix} r & t \\ t & r \end{pmatrix} = \mathrm{e}^{i\gamma} \begin{pmatrix} -i\sqrt{1-g}\mathrm{e}^{i\alpha} & \sqrt{g}\mathrm{e}^{-i\phi} \\ \sqrt{g}\mathrm{e}^{i\phi} & -i\sqrt{1-g}\mathrm{e}^{-i\alpha} \end{pmatrix} \tag{1}$$

with

$$v_E \equiv (2E/\mathsf{m})^{1/2} \tag{2}$$
$$u_E \equiv \frac{u}{\hbar v_E} \tag{3}$$
$$t = \frac{1}{1+iu_E} \tag{4}$$
$$r = t - 1 \tag{5}$$
$$\gamma = \arg(t) = -\arctan(u_E) \tag{6}$$
$$g = \frac{1}{1+u_E^2} = (\cos(\gamma))^2 \tag{7}$$
$$\alpha = 0 \tag{8}$$
$$\phi = 0 \tag{9}$$

Note that we use the common ad-hoc convention of writing the channel functions as $\psi(r) = A\exp(-ikr) + B\exp(-ikr)$ where $r = |x|$. In the alternate convention, which is used for s-scattering, $B \mapsto -B$, so as to have $\boldsymbol{S} = \mathbf{1}$ in the limit of zero coupling ($u = \infty$).