E2122: Bound state in a site coupled to a 1-D conductor

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The problem:

In this problem we find the energy of the bound state E of a particle in a system of one state coupled with a 1-D conductor.
The particle has a mass M. The conductor is infinite: $x \in \{-\infty, \infty\}$.
The site to which the particle is bound is near the point x_0.
If the site was decoupled from the conductor, the particle's bound energy was E_0.
In practice, the site is coupled to the conductor, and the hamiltonian is

$$H = \frac{p^2}{2M} + |0\rangle E_0 \langle 0| + |x_0\rangle \lambda \langle 0| + |0\rangle \lambda \langle x_0|$$

The standard representation of the wave function in the x basis is $\Psi \to (\psi_0, \psi(x))$.
Where the amplitudes are $\psi_0 = \langle 0|\Psi\rangle$, $\psi(x) = \langle x|\Psi\rangle$.
With the normalization $|\psi_0|^2 + \int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$.

(1) Write $\langle 0|H|\Psi\rangle$ in the standard representation.

(2) Write $\langle x|H|\Psi\rangle$ in the standard representation.

Write the equations $H|\Psi\rangle = E|\Psi\rangle$ in the special representation.
Tip: project the equation on the states $\langle 0|$, $\langle x|$ and use the results from previous clauses.
Show that we get a closed equation for $\psi(x)$ with an effective potential of

$$V(x; E) = u(E)\delta(x - x_0)$$

Notice that at this point we treat the bound energy E as a known constant.
Tip: without loss of generality, you can assume the location x_0 is the origin.

(3) Write the expression for $u(E)$.

(4) Write an equation for the bound energy E.

(5) Solve the equation for the case $E_0 = 0$.

The solution:

(1)

$$\langle 0|H|\Psi\rangle = E_0\psi_0 + \lambda\psi(x_0)$$

(2)

$$\langle x|H|\Psi\rangle = -\frac{\psi''(x)}{2M} + \delta(x - x_0)\lambda\psi_0$$

(3)

$$\langle 0|H|\Psi\rangle = E_0\psi_0 + \lambda\psi(0) = E\psi_0 = \langle 0|E|\Psi\rangle$$

$$\langle x|H|\Psi\rangle = -\frac{\psi''(x)}{2M} + \delta(x)\lambda\psi_0 = E\psi(x) = \langle x|E|\Psi\rangle$$

We can isolate ψ_0 from the first equation and substitute it in the second equation, and get

$$-\frac{\psi''(x)}{2M} + \frac{\lambda^2}{E-E_0}\delta(x)\psi(0) = E\psi(x)$$

Therefore

$$V(x;E) = \frac{\lambda^2}{E-E_0}\delta(x) = u(X)\delta(x)$$

(4) $\psi(x)$ will be of the form

$$\psi(x) = \begin{cases} Ae^{-kx} & \text{if } x \geq 0 \\ Ae^{kx} & \text{if } x < 0 \end{cases}$$

For $x \neq 0$:

$$-\frac{k^2}{2M} = E \Rightarrow k = \sqrt{-2ME}$$

For $x = 0$:

$$\psi'(0^+) - \psi'(0^-) = 2M\frac{\lambda^2}{E-E_0}\psi(0)$$

Substituting k with its value from the first equation, we get

$$\sqrt{-2ME} = \frac{M\lambda^2}{E_0 - E}$$

(5) For $E_0 = 0$:

$$E = -\sqrt[3]{\frac{M\lambda^4}{2}}$$