E212: Particle in an potential well, with scatterer

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The problem:

A particle is placed inside an infinite square well [0,2d] in the center there is a delta scatterer $V(x) = -u\delta(x-d)$:

(1) What is the minimum u for which the energy states are negative?
(2) How does the wave function look when we decrease u to a point where the ground state E=0?

The solution:

The time independent schrodinger equation is as follows:

$$[\frac{\hat{p}^2}{2m} + V(x)]\psi(x) = E\psi(x)$$

inserting $\hat{p} = -i\frac{d}{dx}$ and the potential we get:

$$[-\frac{1}{2m}\nabla^2 - u\delta(x-d)]\psi(x) = E\psi(x)$$

with the following boundry conditions for the infinite walls and delta stitching:

$$[1]\psi|_{x=0} = \psi|_{x=2d} = 0$$

$$[2]\psi|_{x=d-} = \psi|_{x=d+}$$

by means of integrating the Schrodinger equation over infinitysimal ϵ around x=d we get the a second stitching condition:

$$[3]\frac{d\psi}{dx}|_{x=d+} - \frac{d\psi}{dx}|_{x=d-} = 2mu\psi|_{x=d}$$

using the equation and the boundry conditions we can introduce a wave function as follows (already satisfying condition [1]):

$$[0,d] \quad \psi(x) = \begin{cases} A\sin(kx) & \text{for} \quad E > 0 \\ A\sinh(ax) & \text{for} \quad E < 0 \end{cases}$$

$$[d,2d] \quad \psi(x) = \begin{cases} A\sin(k(2d-x)) & \text{for} \quad E > 0 \\ A\sinh(a(2d-x)) & \text{for} \quad E < 0 \end{cases}$$

from the scrodinger equation we get:

$$k = \sqrt{2mE} \ for \ E > 0$$

$$a = \sqrt{-2mE} \ for \ E < 0$$

using the stitching conditions:

$$\begin{cases} Ak\cos(kd) + Ak\cos(kd) = 2muA\sin(kd) & \text{for} \quad E > 0 \\ Aa\cosh(ad) + Aa\cosh(ad) = 2muA\sinh(ad) & \text{for} \quad E < 0 \end{cases}$$

or finally:

$$\begin{cases} k\cot(kd) = mu & \text{for} \quad E > 0 \\ a\coth(ad) = mu & \text{for} \quad E < 0 \end{cases}$$

(1) we use the second relation to find the minimum for which E is still negative which means $E \to 0-$. we remember:

$$E = -\frac{a^2}{2m}$$

if we take E to zero we get:

$$\sqrt{2mE}\coth(\sqrt{2mE}d) = mu$$

reminded that:

$$\coth(cx) = \frac{1}{\tanh(cx)} \to \frac{1}{cx}$$

we get:

$$\sqrt{2mE}\frac{1}{\sqrt{2mE}d} = mu \Rightarrow u_0 = \frac{1}{md}$$

So energy levels will remain negative for $u > \frac{1}{md}$

(2) Now let us take the limit [a goes to zero] for the wave function:

$$E < 0 \quad \psi(x) = \begin{cases} lim_{a\to 0} A\sinh(ax) & \text{for} \quad [0,d] \\ lim_{a\to 0} A\sinh(a(2d-x)) & \text{for} \quad [d,2d] \end{cases}$$

reminded that:

$$\sinh(cx) \to cx$$

we get:

$$E < 0 \quad \psi(x) = \begin{cases} \tilde{A}x & \text{for} \quad [0,d] \\ \tilde{A}(2d-x) & \text{for} \quad [d,2d] \end{cases}$$

Another way of getting the wave function at E=0 is by inserting E into the schrodinger equation:

$$[-\frac{1}{2m}\nabla^2 - u\delta(x-d)]\psi(x) = 0$$

This gives rise to the form

$$\psi(x) = \begin{cases} A_1x + B_1 & \text{for} \quad [0,d] \\ A_2x + B_2 & \text{for} \quad [d,2d] \end{cases}$$

Using the same boundery conditions and stitching

$$[1]\psi|_{x=0} = \psi|_{x=2d} = 0$$

$$[2]\psi|_{x=d-} = \psi|_{x=d+}$$

we get:

$$\psi(x) = \begin{cases} Ax & \text{for} \quad [0, d] \\ A(2d - x) & \text{for} \quad [d, 2d] \end{cases}$$

And the delta stitching is automatically satisfied because we specified the u for E=0 (u=1/md):

$$[3]\frac{d\psi}{dx}|_{x=d+} - \frac{d\psi}{dx}|_{x=d-} = 2mu\psi|_{x=d}$$

$$A + A = 2Aumd = 2A$$