Ex1660: Josephson Junction and Josephson Current

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The problem:

A capacitor is composed of two superconducting plates. In a superconductor the charge carriers are "Cuper Pairs" which can be treated as bosons with charge $e^* = 2e$. At a neutral state $|n = 0\rangle$, part of the pairs are placed in the left plate and some are at the right plate. The charges state $|n\rangle$ is gained by transferring n pairs from the left plate to the right plate. The system's base is given by $|n\rangle$ states where $n = integer$. The displacement operator is defined by $D|n\rangle = |n + 1\rangle$.
Assume that the transition amplitude per unit time of a pair from plate to plate by tunneling is Ω.
In addition assume that the capacity of the capacitor in C.

In Sub-question 4 assume that the system is prepared in the base state $|n = 0\rangle$ and use first-order time dependent perturbation theory.
In Sub-questions 5-6 use the classical picture of evolution in phase space.

(1) Define the eigenstate $|\varphi\rangle$ of $\hat{D}$ whith eigenvalue $\exp(-i\varphi)$.
(2) Write the system's Hamiltonuan using conjugate operators of $\hat{n}, \hat{p}$.
(3) Define the current operator $\hat{I}$ as implied by the formula $\langle \hat{I} \rangle = \frac{d\langle \hat{n} \rangle}{dt}$.
(4) What is the probability to find the system after a short period of time at $|n \neq 0\rangle$.
(5) Find the frequency of the system's small oscillations.
(6) In which $|n\rangle$ states it is likely to find the system after a long period of time.

The solution:

(1) Regarding $\hat{\varphi}$ and $\hat{n}$ as an analog to momentum and position, the eigenstate $|\varphi\rangle$ of $\hat{D}$ can be described as a superposition of $|n\rangle$ states :

$$|\varphi\rangle = const \cdot \sum_n e^{i\varphi_n} |n\rangle$$

(2) Consider the energy of a capacitor as $E = \frac{q^2}{2C}$. Since the probability per time unit for transition is the same for both direction, and assuming for simplicity that Ω is real, the Hamiltonian is:

$$\hat{\mathcal{H}} = \frac{(2e \cdot \hat{n})^2}{2C} + \Omega \hat{D} + \Omega \hat{D}^\dagger = \frac{(2e \cdot \hat{n})^2}{2C} + \Omega \left(e^{i\hat{\varphi}} + e^{-i\hat{\varphi}} \right) = \frac{(2e \cdot \hat{n})^2}{2C} + 2\Omega \cos(\hat{\varphi})$$

(3) According to the relation $\langle \hat{I} \rangle = \frac{d\langle \hat{n} \rangle}{dt}$ the current operator is defined by:

$$\hat{I} = i[\hat{\mathcal{H}}, \hat{n}] + \frac{\partial n}{\partial t}$$

$\frac{\partial n}{\partial t} = 0$ since $\hat{n}$ is not directly dependent of t.
Hence the current operator is:

$$\hat{I} = i\left[\hat{\mathcal{H}}, \hat{n}\right] = i\left[\Omega \hat{D} + \Omega \hat{D}^\dagger, \hat{n}\right] = i\Omega(\hat{D} - \hat{D}^\dagger) = 2\Omega \sin \hat{\varphi}$$

(4) After a short period of time, only the first transition is permitted. Therefor using first-order time dependent perturbation theory the probability is:

$$P_t(n|0) = |W_{10}|^2 \cdot \left| \frac{1 - e^{i(E_1 - E_0)t}}{E_1 - E_0} \right|^2 = |\Omega|^2 \cdot \left| \frac{1 - e^{i\left(\frac{(2e)^2}{2C} - 0\right)t}}{\frac{(2e)^2}{2C} - 0} \right|^2 = \frac{\Omega^2 C^4}{e^4} \sin^2\left(\frac{e^2}{C} t\right)$$

(5) In order to find the frequency of small oscillations there is a need to compute a second-order Taylor series expansion around it's minimum, which is at π. There is no need to considerate -2Ω because the Hamiltonian is gauge invariant.

$$\hat{\mathcal{H}} \approx \frac{(2e \cdot \hat{n})^2}{2C} + \Omega\,(\hat{\varphi} - \pi)^2$$

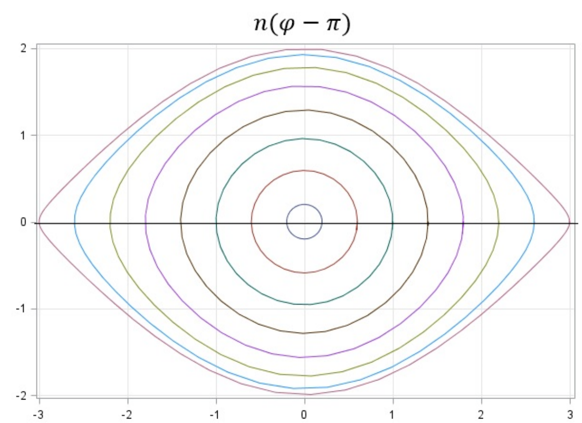
Observing the Hamiltonian of a classical oscillator $\hat{\mathcal{H}} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$ the following two relations are seen:

$$[a] \quad \frac{4e^2}{2C} \to \frac{1}{2m} \qquad\qquad [b] \quad \Omega \to \frac{1}{2} m\omega^2$$

With some algebra the frequency is obtained:

$$\omega = \sqrt[2]{\frac{8e^2\Omega}{C}}$$

(6)The classical picture of evolution in phase space of an oscillator is described by a separatrix:



Note that the lines in the figure describe the phase space when the levels of E are constant, the type of resulting curve, depends upon the value of the energy.
In the beginning the system is at $|n = 0\rangle$ therefor the states are spread in phase space as a line $n = 0$. The energy in this configuration is $E_{n=0} = 2\Omega \cos\varphi$, it is easy to see that the maximal energy is $E = 2\Omega$ (when $\varphi = 0$). After evolution in time all the states in the separatrix can be achieved. At this time the highest energy is the energy of the highest state, therefor, because the energy is conserved, it is equal to $E_{n=0}$:

$$E_{n_{max}} = \frac{2e^2 n_{max}^2}{C} + 2\Omega \cos\pi = 2\Omega$$

Therefore, the possible states after a long time are:

$$|n| \le \sqrt{\frac{2\Omega C}{e^2}}$$