E1660: Josephson Junction and Josephson Current

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The problem:

A capacitor is composed of two superconducting plates. In a superconductor the charge carriers are "Cuper Pairs" which can be treated as boson with charge $e^* = 2e$. At a neutral state $|n = 0\rangle$, part of the pairs are placed in the left plate and some are at the right plate. The charged state $|n\rangle$ is gained by transferring n pairs from the left plate to the right plate. The system's base is given by $|n\rangle$ states where $n = integer$. The displacement operator is defined by $D|n\rangle = |n+1\rangle$.
Assume that the transition amplitude (per unit time) of a pair from plate to plate (by tunneling) is Ω. In addition assume that the capacity of the capacitor is C.
In question 4 assume that system is prepared in state $|n = 0\rangle$ and use first-order perturbation theory.
In question 5-6 use the classical picture of evolution in phase space.

(1) Define the eigenstate $|\varphi\rangle$ of D with eigenvalue $\exp(-i\varphi)$.
(2) Write the system's Hamiltonian using conjugate operators of n, φ.
(3) Define the current operator I as $\frac{d<n>}{dt}$.
(4) What is the probability to find the system after a short period of time at $|n \neq 0\rangle$ state.
(5) Find the frequency of the system's small oscillations.
(6) In which $|n\rangle$ states it is likely to find the system after a long period of time.

The solution:

(1) The eigen state $|\varphi\rangle$ of D can be described as a superposition of $|n\rangle$ states:

$$|\varphi\rangle = const \cdot \sum_n \exp(i\varphi_n)|n\rangle$$

(2) Since the probability per time unit for transition is the same for both directions The Hamiltonian is given by:

$$H = \frac{(2e \cdot \hat{n})^2}{2C} + \Omega(\hat{D} + \hat{D}^\dagger) = \frac{(2e \cdot \hat{n})^2}{2C} + 2\Omega\cos(\hat{\varphi})$$

(3) According to Ehrenfest theorem:

$$\frac{d < n >}{dt} = \frac{1}{i} < [n, H] > + < \frac{\partial n}{\partial t} >$$

The second term is zero and we get:

$$-i < [n, H] >= \frac{1}{i}\Omega([\hat{n}, \hat{D} + \hat{D}^\dagger]) = \frac{1}{i}\Omega(\hat{D} - \hat{D}^\dagger) = 2\Omega\sin(\varphi)$$

(4) After a short period of time only the first transition is allowed and the probability is:

$$P_t(1|0) = |W_{10}|^2 |\frac{1 - e^{i(E_1 - E_0)t}}{E_1 - E_0}|^2 = |\Omega|^2 |\frac{1 - e^{i(\frac{(2e)^2}{2C} - 0)t}}{\frac{(2e)^2}{2C} - 0}|^2 = \frac{\Omega^2 C^4}{e^4}\sin^2(\frac{e^2}{C}t)$$

(5) In order to find the frequency of the small oscillations we need to Taylor expand the potential term up to second order near it's minimum at π:

$$E = \frac{(2e \cdot n)^2}{2C} + 2\Omega\cos(\varphi) \approx \frac{(2e \cdot n)^2}{2C} - 2\Omega + \Omega(\varphi - \pi)^2$$

And the frequency is therefore:

$$\omega = \sqrt{\frac{8e^2\Omega}{C}}$$

(6) At the beginning the system is at $|n = 0\rangle$ so the states are spread in phase space $(n(\varphi))$ as a line at $n = 0$, after evolution in time the states fill the space inside the separatrix. The separatrix is the maximal ellipse that can be achieved in the phase space classically. So we get the condition for n:

$$\frac{2e^2 \cdot n^2}{C} = 2\Omega$$

and we get:

$$|n| \leqslant \sqrt{\frac{\Omega C}{e^2}}$$